# CHAPTER 2

# DERIVATION OF STATE EQUATIONS AND PARAMETER DETERMINATION OF AN IPM MACHINE

## 2.1 Derivation of Machine Equations

A model of a 3 phase PM machine is shown in Figure 2.1. Both the abc and the dq axes are shown in the figure. The magnetizing current, due to the presence of the magnet, is represented in Figure 2.1 by the current source $i_f$ . The fictitious current source ( along with the associated field winding $L_{md}$) will be used in the derivation to follow to obtain an expression for the flux which is equivalent to the flux created by the presence of the magnet

It is assumed in the following derivation that the rotor speed of the machine is constant . Thus, no current flows in the damper windings contained within the rotor and their presence may be ignored. It is also assumed that the machine is balanced.

The stator voltage equations of the IPM are [31]

$$\begin{aligned} v_a &= \frac{d}{dt}\lambda_a + i_a r_1 \\ v_b &= \frac{d}{dt}\lambda_b + i_b r_1 \\ v_c &= \frac{d}{dt}\lambda_c + i_c r_1 \ , \end{aligned} \tag{2.1}$$

where $v_a$ , $v_b$ , $v_c$ are the a,b,c stator terminal voltages , $\lambda_a$ , $\lambda_b$ ,$\lambda_c$ are the flux linkages in the abc plane, $i_a$ , $i_b$ ,$i_c$ are the abc stator currents, and $r_1$ is the stator resistance of each phase

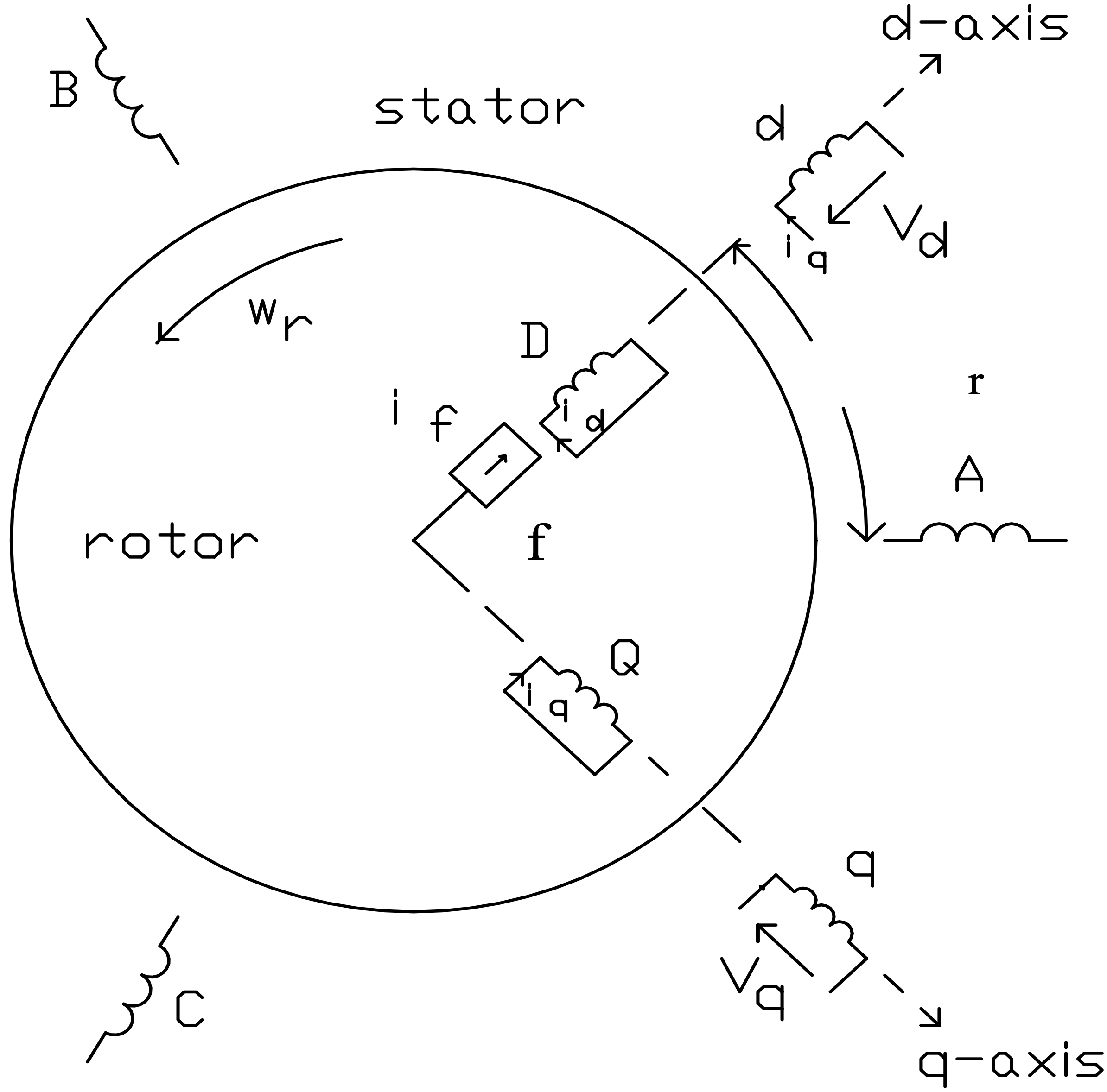


Figure 2.1. Model of a brushless PM synchronous machine

of the stator winding. Thus, the assumption has been made that the resistances for each phase of the stator are the same.

The flux linkages expressed in terms of the stator and terms of the current are

$$= \begin{matrix} L & L & L \\ L & L & L \\ L & L & L \end{matrix} \quad \begin{matrix} i \\ i \\ i \end{matrix} + \begin{matrix} L & 0 & 0 \\ L & 0 & 0 \\ L & 0 & 0 \end{matrix} \quad \begin{matrix} i \\ 0 \\ 0 \end{matrix} .$$

The inductances given in Equation (2.2) will be kept in their symbolic form, transformed into their equivalent dq axis counterparts, and then be defined in terms of the physical construction of machine.

Due to the salient nature of the rotor, the inductances of Equation (2.2) are functions of the position of the rotor and (assuming that the rotor is spinning) are functions of time. This means that the inductance parameters are constantly changing - making the analysis of the machine very difficult in its present form. A transformation commonly referred to as Park's transformation allows the equations describing the machine to be transformed into a reference frame where the inductances are not functions of time. The reference frame chosen to accomplish this goal is dependent upon the type of machine being looked at. For a synchronous machine, this reference frame is that of the rotor. In other words, the stator voltages, currents, and inductances will be projected onto the rotor side of the machine and, thus, be rotating at the same speed with which the rotor is spinning. With this accomplished, the inductances no longer vary with the position of the rotor.

The transformation of a 3 phase balanced voltage, current, or flux linkage from the

$$h_{qdo} = T(\theta)\, h_{abcs} \ , \qquad (2.3)$$

abc to the dq reference plane may be expressed as [32]

$$T(\quad) = \frac{2}{3} \begin{matrix} (\quad) & (\quad - \frac{2}{3}) & (\quad + \frac{2}{3}) \\ (\theta) & (\theta - \frac{2\pi}{3}) & (\theta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{matrix} \ ,$$

where

and θ denotes the reference frame chosen. For the rotor reference frame, $\theta = \theta_r$ . Expressing the q d and o terms individually and substituting $\theta = \theta_r$ gives

where, for balanced conditions, $h_0 = 0$ and the o term will, from this point forward, be ignored. So, using Equation (2.4), the flux linkages may be expressed in the dq plane as

Taking the derivative of $\lambda_q$ in Equation (2.5) with respect to time gives

$$\frac{d\lambda_a}{dt}\cos(\theta_r) + \frac{d\lambda_b}{dt}\cos(\theta_r - \frac{2\pi}{3}) + \frac{d\lambda_c}{dt}\cos(\theta_r + \frac{2\pi}{3}) =$$
$$[v_a\cos(\theta_r) + v_b\cos(\theta_r - \frac{2\pi}{3}) + v_c\cos(\theta_r + \frac{2\pi}{3})] -$$
$$[i_a r_1\cos(\theta_r) + i_b r_1\cos(\theta_r - \frac{2\pi}{3}) + i_c r_1\cos(\theta_r + \frac{2\pi}{3})]$$
$$= v_{qs} - i_{qs} r_1 \qquad (2.9)$$

Substituting the results of Equations (2.7) and (2.9) into (2.6) gives

$$\frac{d\lambda_q}{dt} = -\lambda_d\frac{d\theta_r}{dt} + v_{qs} - i_{qs} r_1 \ . \qquad (2.10)$$

Letting

$$p = \frac{d}{dt}\ ,\ and\ \omega_r = \frac{d\theta_r}{dt}\ ,$$

then

$$p\lambda_q = -\lambda_d\omega_r + v_{qs} - i_{qs} r_1 \ . \qquad (2.11)$$

Therefore, the q-axis voltage can be written as

$$v_{qs} = p\lambda_q + i_{qs} r_1 + \lambda_d\omega_r \ . \qquad (2.12)$$

Similarly, taking the derivative of the d-axis flux linkage of Equation (2.5) gives

After rearranging Equation (2.16) and substituting Equation (2.10) into it, the final result for the d axis voltage is

$$v_{ds} = p\lambda_d + i_{ds} r_1 - \lambda_q \omega_r \, . \tag{2.17}$$

In order to obtain the individual self and mutual inductance terms ( given in Equation (2.2 )), the parameters may be defined in terms of the physical construction of the machine , i.e. number of windings, physical dimensions, etc. , and then the equations obtained may be transformed into the dq reference frame. Alternatively, as was done in this thesis, the flux linkage equations may be transformed directly into the dq rotor reference frame ( $\theta$ equals $\theta_r$) and the q and d axis inductances may be defined directly in that frame of reference. Thus,

$$\begin{bmatrix} \lambda_q \\ \lambda_d \\ \lambda_o \end{bmatrix} = T(\theta) \begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{cs} \end{bmatrix} = T(\theta) \begin{bmatrix} L_{aa} & L_{ab} & L_{ac} \\ L_{ba} & L_{bb} & L_{bc} \\ L_{ca} & L_{cb} & L_{cc} \end{bmatrix} (T(\theta))^{-1} \begin{bmatrix} i_{qs} \\ i_{ds} \\ i_o \end{bmatrix} + T(\theta) \begin{bmatrix} L_{afd} & L_{adr} & L_{aqr} \\ L_{bfd} & L_{bdr} & L_{bqr} \\ L_{cfd} & L_{cdr} & L_{cqr} \end{bmatrix} \begin{bmatrix} i_f \\ i_{dr} \\ i_{qr} \end{bmatrix} , \tag{2.18}$$

where

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The d and q axis mutual inductances are given as [12]

$$L_{mq} = C_q \, L_m$$
$$L_{md} = C_d \, L_m \ , \tag{2.20}$$

where $L_m$ is the inductance of a machine with a uniform air gap and no magnets. This inductance is determined from the flux linking with $N_1 C_w$ effective turns, and is given as

$$L_m = 1.273\, \mu_o \, m_1 \left(\frac{N_1 C_w}{P}\right)^2 \frac{D_i L}{g} 10^{-8}\, H \ , \tag{2.21}$$

where all the terms given in Equation (2.21) are in meter, kilogram, second (MKS) units

$\mu_o$ = permeability of free space

$m_1$ = number of phases of the machine

$N_i$ = number of series turns per phase

$C_w$ = a winding factor which is a product of the distribution and pitch factors

$D_i$ = stator inner diameter

P = number of poles

L = core length

g = effective air gap length.

$C_q$ and $C_d$ are factors which account for the presence of the magnets and are, for an interior permanent magnet, given as

$$C_q = \rho - \sin\frac{\rho\pi}{\pi}$$

$$C_d = \rho + \sin\frac{\rho\pi}{\pi} - \frac{(\frac{8}{\rho}\pi^2)\sin^2\rho\frac{\pi}{2}}{1+\frac{R_g}{R_m}} \quad , \qquad (2.22)$$

where

ρ = the pole arc

$R_g$ = reluctance of the air gap

$R_m$ = reluctance of the magnet .

The open circuit magnet flux $\lambda_e$ for an IPM machine is given as [12]

$$\lambda_e = \frac{4.44}{2\pi} N_i C_w \frac{(\pi D_i L)}{P} B^!_f * 10^{-8} \quad , \qquad (2.23)$$

where $B^!_f$ is the amplitude of the fundamental flux density created by an individual magnet.

In summary, the voltage and flux equations needed to analyze a permanent magnet under the stated assumptions are

$$v_{qs} = p\,\lambda_{qs} + i_{qs} r_s + \lambda_{ds}\,\omega$$
$$v_{ds} = p\,\lambda_{ds} + i_{ds} r_s - \lambda_{qs}\,\omega \quad , \qquad (2.24)$$

where

$$\lambda_{ds} = L_{ds}\, i_{ds} + \lambda_e$$
$$\lambda_{qs} = L_{qs}\, i_{qs} \quad ,$$

and the subscript s on the q and d axis stator terms has been added, $r_s$ has replaced $r_1$ as the symbol used to represent the stator resistance, and the subscript r has been dropped from ω. Under steady-state, the derivatives of the state variables of Equation (2.24) are zero so, at steady state the voltage equations may be written as

$$V_{qs} = I_{qs} r_s + \lambda_{ds} \omega$$

$$V_{ds} = I_{ds} r_s - \lambda_{qs} \omega \quad . \tag{2.25}$$

A d-q axis schematic diagram representing the equations given in (2.25) is shown in Figure 2.2

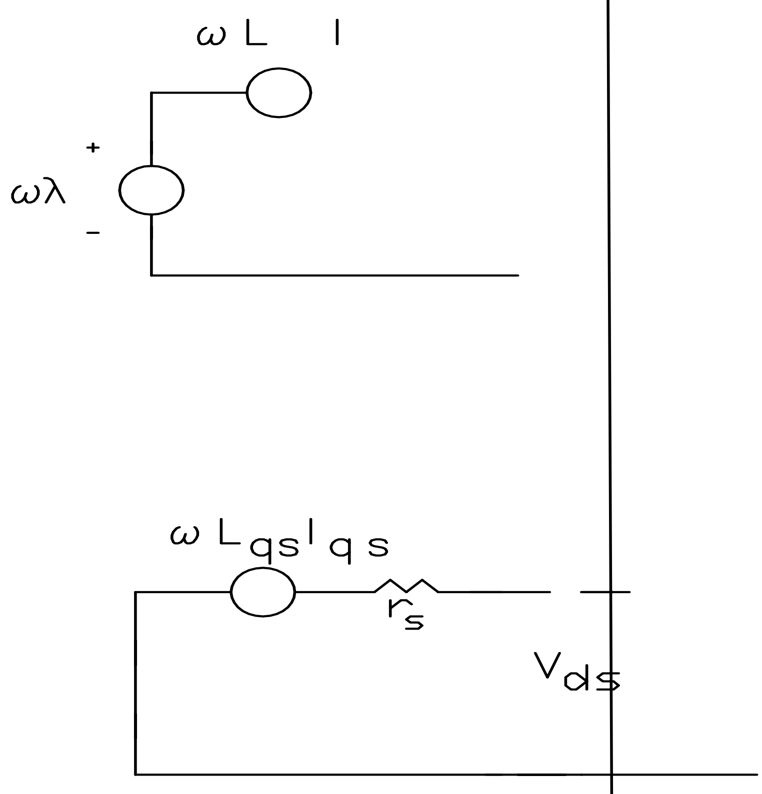


Figure 2.2. Schematic diagram representing the steady state q and d axis voltage equations of a PM machine

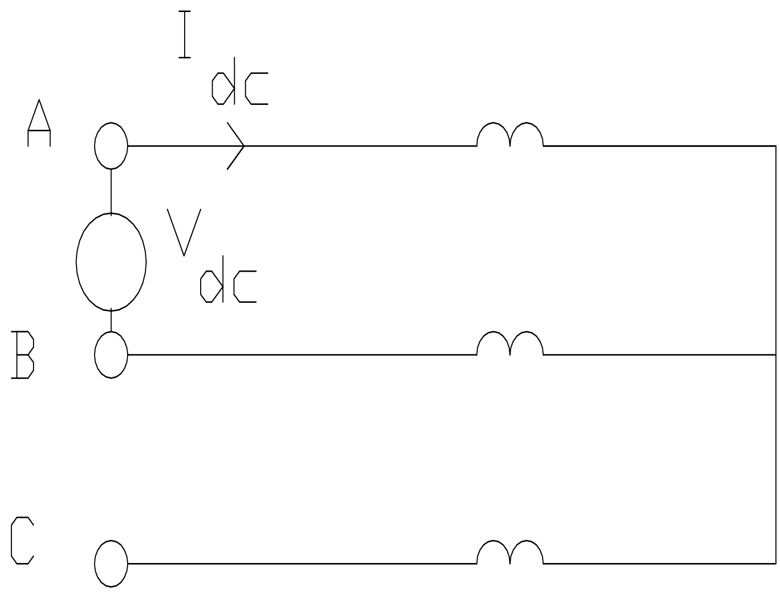


Figure 2.3. Schematic diagram of dc test used to determine stator resistance

The stator resistive value $r_s$ was found by applying a dc voltage across two terminals of the stator and measuring both the voltage and the current which flowed through the terminals ( see Figure 2.3). The stator resistance for a single phase is given as

$$r_s = \frac{V_{dc}}{2\, I_{dc}} \ . \tag{2.27}$$

The voltages and currents in Equation (2.26) were found by varying a three phase balanced resistive load from a high value to a low value and recording the terminal voltage and the current output from the generator. The measurements made were the line to line voltages and the phase currents. In order to convert the voltages and currents into their dq components, the torque angle δ was needed. The power factor of the machine is also needed in parameter determination, but, since the generator was feeding a resistive load, the current out of the generator was in phase with the voltage at the terminals, so the power factor was unity. The torque angle was found by measuring the difference in angle of the voltages of a search coil located across phase "a" of the stator and the terminal voltage appearing at the stator terminal of phase "a." This method was not an ideal way to measure the torque angle because the oscilloscope used to measure the angle between the two voltages gave a varying readout even though the load and speed of the generator were constant. An average of the numbers was taken and used as the torque angle. It would have been much easier (and probably more accurate) to have a commercially available torque angle measuring device; however, no such device was available. Nevertheless, the strong corroboration between measured and predicted results suggests that the method used was an acceptable means of obtaining the torque angle.

Once the stator voltages, currents, and torque angle are known, the dq voltages and

$$\begin{aligned} V_{qs} &= V_s \operatorname{Cos}(\delta) \\ V_{ds} &= -V_s \operatorname{Sin}(\delta) \\ I_{qs} &= I_s \operatorname{Cos}(\gamma) \\ I_{ds} &= -I_s \operatorname{Sin}(\gamma)\ , \end{aligned} \tag{2.28}$$

currents can be found by the following relations:

where $V_s$ is the peak line to neutral voltage, $I_s$ is the peak stator current, and γ is the sum of the torque angle δ and the power factor angle θ. Since the power factor is unity (since the generator is feeding a purely resistive load), then γ is equal to δ.

With the dq voltages and currents and the stator resistance known, the inductance in the q axis can easily be found and is given as

The inductance in the d axis and the magnet flux linkage are not as easy to find as the q axis inductance. The two terms are contained in the same equation and are, in a sense, coupled together.

One method of finding the magnet flux involves running a no load test on the PM machine for a range of frequencies and measuring the terminal voltage of the machine and the voltage across the terminals of the search coil. An empirical relationship between the search coil voltage and the magnet voltage (and thus the magnet flux linkage) can be developed since, at a no load condition, the terminal voltage of the machine is equal to the magnet voltage. Figure 2.4 shows a plot of the rms voltage of the magnet vs. the air gap voltage for both the series connection (high voltage) and the parallel connection (low voltage) of the stator winding of the PM machine.

The low voltage connection was not used in

The

Figure 2.4. Measured line to neutral rms terminal generator voltage vs

Hertz since most loads were designed to operate at or near that particular voltage.

After the magnet flux term has been determined (by $\lambda_e = E_o / \omega$), then $L_{ds}$ can be found by

$$L_{ds} = \frac{V_{qs} - r_s I_{qs} - \omega \lambda_e}{\omega I_{ds}} \quad . \tag{2.31}$$

The plots of $L_{qs}$, $L_{ds}$, and $\lambda_e$ are given in Figures 2.5-2.8. The empirical relationships of the parameters as a function of peak stator current $I_s$ are given as

$$\begin{aligned} \ln(\frac{1}{L_{qs}}) &= -.0013\, I_s^4 + .0293\, I_s^3 - .2303\, I_s^2 + .8684\, I_s + .790 \\ \ln(\frac{1}{L_{ds}}) &= -.0011\, I_s^4 + .0251\, I_s^3 - .210\, I_s^2 + .9096\, I_s + 1.505 \\ \lambda_e &= .0002\, I_s^3 - .0041\, I_s^2 + .0208\, I_s + .1863 \ . \end{aligned} \tag{2.32}$$

In a conventional synchronous machine with a field winding, the d axis inductance is larger than the q axis inductance; however, comparing the magnitudes of $L_{qs}$ and $L_{ds}$ in Figures 2.5 and 2.6, it can be seen that $L_{qs}$ is larger. This phenomenon, called inverse saliency, is caused by the magnet depth appearing as basically an air gap in the d-axis. As the machine is loaded, it can also be seen that the magnet voltage $E_o$ first increases and then decreases. The initial increase in magnet voltage is due to the fact that, at very light loads, the bridge becomes highly saturated and much of the magnet flux flows through it and does not contribute towards a useful air gap voltage.

It was mentioned earlier in this section that the reason the parameters were made functions of the peak stator current and not the total flux linkage was that the when the

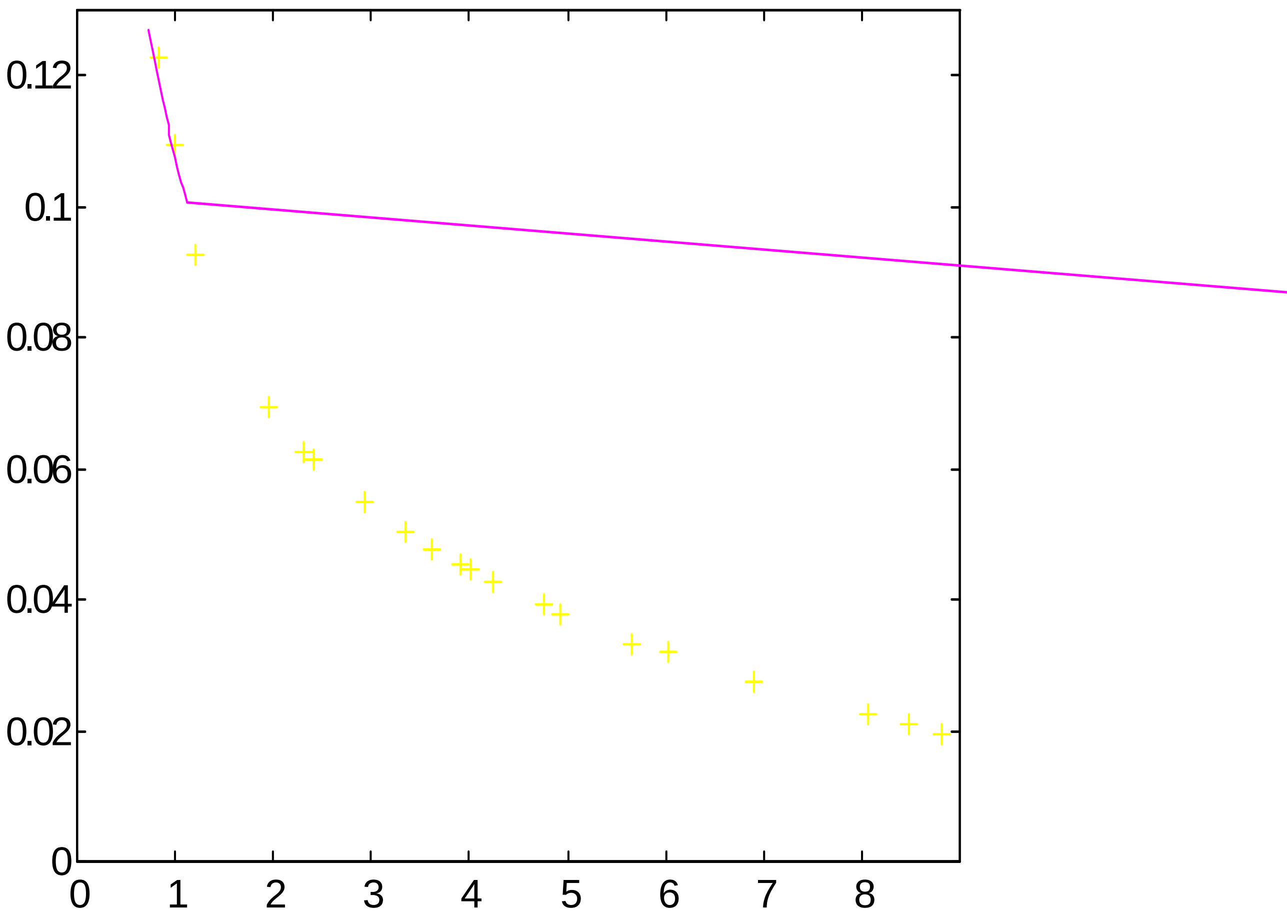


Figure 2.6. Measured d axis inductance vs peak stator current

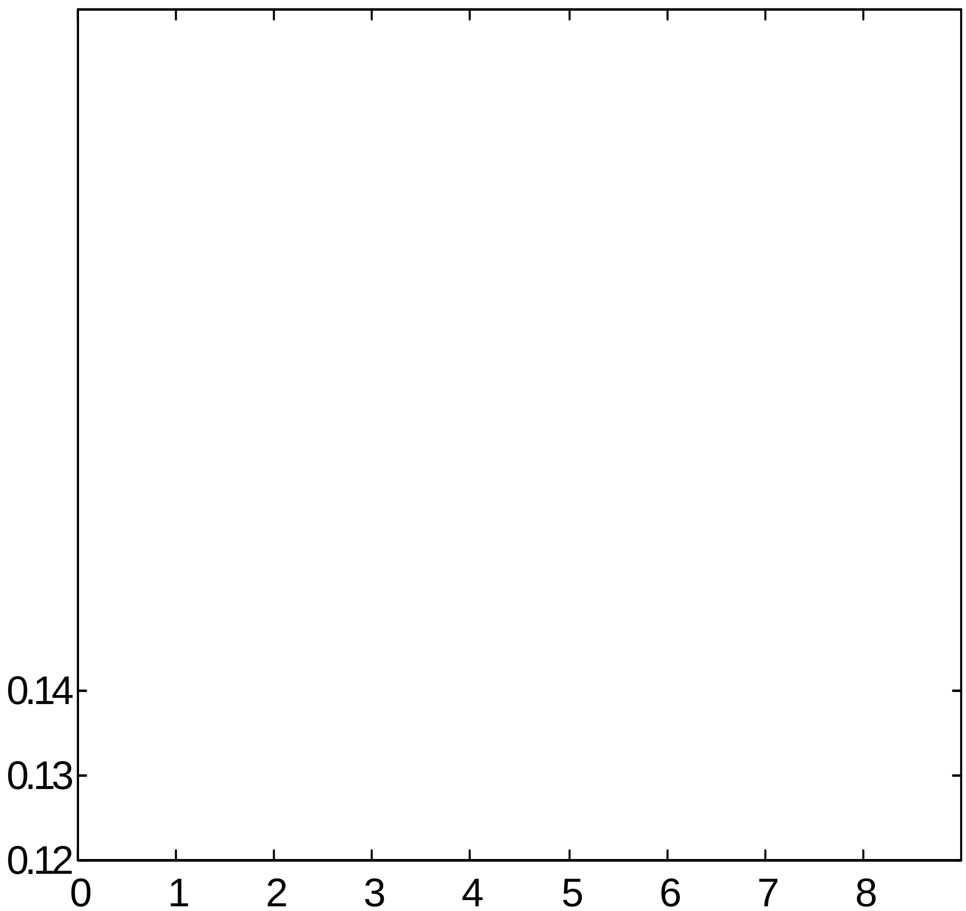


Figure 2.7. Measured magnetic flux vs peak stator current

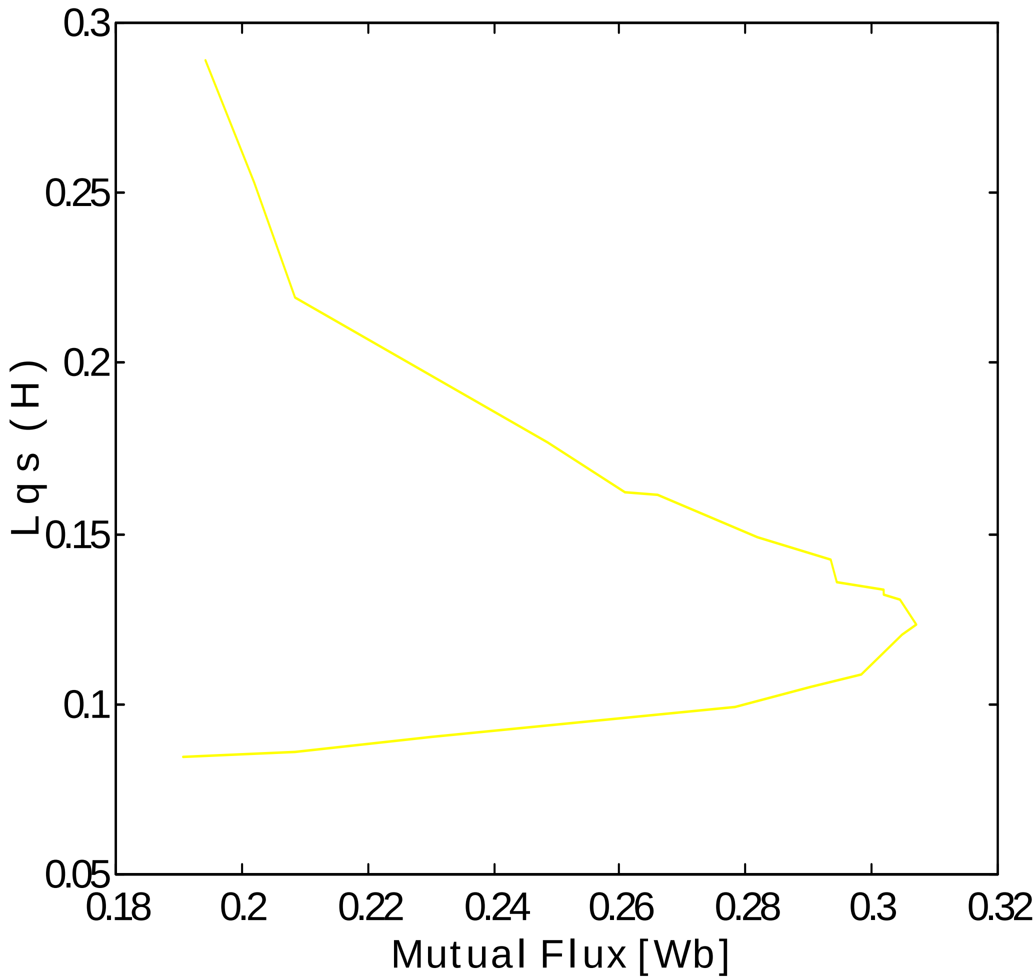


Figure 2.8. Measured q axis inductance vs peak mutual flux

parameters were plotted as a function of the total mutual flux linkage $\lambda_{mm}$, there were regions in which two possible solutions exist. An example of this is shown in Figure 2.8 where the q axis inductance is plotted vs the mutual flux linkage. It can be seen from the figure why determining $L_{qs}$ from a given value of $\lambda_{mm}$ would be difficult. For example, if $\lambda_{mm}$ was given as 0.28 Wb, then $L_{qs}$ could either be 0.1 or 0.15 H. The same problem was present for the parameters $L_{ds}$ and $\lambda_e$ as well.