DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

GRADED CLIFFORD ALGEBRAS AND GRADED SKEW CLIFFORD ALGEBRAS: A SURVEY OF THE ROLE OF THESE ALGEBRAS IN THE CLASSIFICATION OF

that would be as successful as commutative algebraic geometry had been for commutative algebra, M. Artin and W. Schelter [AS] introduced the notion of a regular algebra. A few years later Artin, Tate and Van den Bergh [AS, ATV1, ATV2] went on to classify the generic classes of regular algebras of global dimension three. The main idea behind this classification, introduced by Artin, Tate and Van den Bergh in [AS, ATV1, ATV2], involved using certain graded modules in place of geometric data, for example, point modules in place of certain points and line modules in place of certain lines. In particular, Artin, Tate, and Van den Bergh showed that such algebras could be associated to certain subschemes E (typically of dimension one) of $\mathbb{P}^2$ where points in the scheme E parametrize certain modules over these algebras called point modules. The technique involved the definition of a quantum analog of the projective plane $\mathbb{P}^2$

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Example. Suppose $M_1 = \; {}^{2}\;{}_{0}$ and $M_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, where $2 \Bbbk$. Let C be the graded $\Bbbk$-algebra on degree-one generators $x_1; x_2$ and on degree-two generators $y_1; y_2$. Using the relations in Definition 2.1(i), we obtain $2\, x_1^2 = 2\, y_1$, $x_2^2 = y_2$, and $x_1x_2 + x_2x_1 = \; y_1 = \; x_1^2$. Thus, $\frac{\Bbbk\langle x_1; x_2\rangle}{\langle x_1x_2 + x_2x_1 \quad x_1^2\rangle}$ maps onto C.

Definition 2.3. [AS] Let $A = \; {}^{1}_{i=0} A_i$ be a finitely generated, N-graded, connected $\Bbbk$-algebra. The algebra A will be called regular (or AS-regular) if it satisfies the following properties:

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Results in [AL, Lb] relates GCAs to regular algebras using the geometry we discuss above.

Proposition 2.6. [AL, Lb] *The graded Clifford algebra C is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series* $1/(1-t)^n$ *if and only if the associated quadric system is base point free; in this case, C is Artin-Schelter regular and is a noetherian domain.*

Example. Let M_1 and M_2 be as in the previous example. Since the quadratic forms q_1 and q_2 associated to M_1 and M_2, respectively, are base point free, then the $\Bbbk$-algebra $\frac{\Bbbk\langle x_1, x_2\rangle}{}$

of intersection points of two planar cubic divisors determines the number of isomorphism classes of point modules over such GCAs. The two planar cubic divisors, in turn, parametrize quadrics of rank at most two. This is where the result that $r_1 + 2r_2$ where r_i refers to the number of quadrics of rank i from Theorem 2.8 comes into play. For this subsection, we will call a linear system of quadrics q, such that $\mathbb{P}(q) = \mathbb{P}^2$, a net of quadrics and a fany

We write $M\ (n; |)$ for the set of -symmetric matrices in $M(n; |)$ and note that if $\ _{ij} = 1$ for all i, j, then $M\ (n; |)$ is the set of all symmetric matrices.

Remark 3.2. For $\{i, j\} \quad \{1, \ldots, n\}$, let

matrices $M_1; M_2$ and M_3 yield a regular GSCA, A, of global dimension three on generators $x_1; x_2; x_3$ with de ning relations

$$x_1x_2 + \quad {}_{12}x_2x_1 = 0$$

Type	Defining Relations	Conditions
H	$y^2 = x^2$, $zy = -iyz$, $yx - xy = iz^2$, where i is a primitive fourth root of unity.	$\mathrm{char}(\Bbbk) \neq 2$
B	$xy + yx = z^2 - y^2$, $xy + yx = az^2 - x^2$, $zx - xz = a(yz - zy)$, $a \in \Bbbk$,	$a(a-1) \neq 0$, $\mathrm{char}(\Bbbk) \notin \{2, 3\}$
A	$axy + byx + cz^2 = 0$, $ayz + bzy + cx^2 = 0$, $azx + bxz + cy^2 = 0$, $a, b, c \in \Bbbk$.	$abc \neq 0$, $\mathrm{char}(\Bbbk) \neq 3$ $(3abc)^3 \neq (-a^3 + b^3 + c^3)^3$, $a^3 = b^3 \neq c^3$, $b^3 = c^3 \neq a^3$,

Remark 3.13. Let : $P(M (n; |)) ! P(S_2)$ be de ned by $(M) = z^T M z$. Hereafter, we x $M_1; \ldots; M_n$ 2 $M (n; |)$. For each $k = 1; \ldots; n$, we x representatives $q_k = (M_k)$. Moreover, if M is a -symmetric matrix in $P(M (n; |))$ and if -rank$(T(M))$ 2, then we de ne -rank(M) to be -rank of (M).

Remark 3.14. If A is a GSCA, then [CaV, Lemma 1.13] implies that y_i 2 $(A_1)^2$ for all $i = 1; \ldots; n$ if and only if $M_1; \ldots; M_n$ are linearly independent. Henceforth, we assume that $M_1; \ldots; M_n$ are linearly independent.

Example.

since the ideal

four and this was the motivation for Section 3. Another goal is to provide candidates for generic quadratic regular algebras of global dimension four so as to contribute towards the classification of quadratic regular algebras of global dimension four. As it were, one of the families of GSCAs discussed in [CaV, §5] is a candidate for generic quadratic regular algebras of global dimension four. As discussed in [V], these algebras should have a point scheme with exactly twenty points and a one-dimensional line scheme.

Using techniques that involve Plücker coordinates, [ChV] analyzes the line scheme of this family of algebras. They find that the line scheme consists of one spatial elliptic curve, four planar elliptic curves, and a subscheme in a $\mathbb{P}^3$ consisting of the union of two nonsingular conics. In [TV], the authors analyze yet another family of algebras with a point scheme consisting of exactly twenty points and a one-dimensional line scheme and their analysis leads to a line scheme consisting of one spatial elliptic curve, one nonplanar rational curve, two planar elliptic curves, and two subschemes that is the union of a nonsingular conic and a line. This lead the authors to conjecture that the line scheme of the most generic quadratic regular algebra of global dimension four should be isomorphic to the union of two spatial elliptic curves and four planar elliptic curves.

Remark 4.1. `Classically' defined Clifford algebras (see [L]) differ from the graded Clifford and graded skew Clifford algebras discussed here. One of the main differences is that the former are $\mathbb{Z}_2$-graded while the latter are $\mathbb{N}$-graded. Work done by [CK] has sought to bridge these differences.

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