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as twisted group algebras $\mathbb{R}^t[(\mathbb{Z}_2)^n]$ of a finite group $(\mathbb{Z}_2)^n$ [5–7, 9, 13, 23]. While this last approach is not pursued here, for a connection between Clifford algebras $C\ell_{p,q}$ and finite groups, see [1, 6, 7, 10, 20, 21, 27] and references therein.

In Section 3, we state the main Structure Theorem on Clifford algebras $C\ell_{p,q}$ and relate it to the general theory of semisimple rings, especially to the Wedderburn-Artin theorem. For details of computation of spinor representations, we refer to [2] where these computations were done in great detail by hand and by using CLIFFORD, a Maple package specifically designed for computing and storing spinor representations of Clifford algebras $C\ell_{p,q}$ for $n = p + q \leq 9$ [3, 4].

Our standard references on the theory of modules, semisimple rings and their representation is [26]; for Clifford algebras we use [11, 18, 19] and references therein; on representation theory of finite groups we refer to [17, 25] and for the group theory we refer to [12, 14, 22, 26].

2. Introduction to Semisimple Rings and Modules

k-algebra A is both *left and right artinian*, that is, any descending chain of left and right ideals stops (the DCC *ascending chain condition*).

Thus, every Clifford algebra $C\ell_{p,q}$, as well as every group algebra kG, when G is a finite group, which then makes kG finite dimensional, have both chain conditions by a dimensionality argument.

Definition 2. A left ideal L in a ring R is a *minimal left ideal* if $L \neq (0)$ and there is no left ideal J with $(0) \subsetneq J \subsetneq L$.

One standard example of minimal left ideals in matrix algebras $R = \mathrm{Mat}(n,k)$ are the subspaces $\mathrm{COL}(j)$, $1 \leq j \leq n$, of $\mathrm{Mat}(n,k)$ consisting of matrices $[a_{i,j}]$ such that $a_{i,k} = 0$ when $k \neq j$ (cf. [26, Example 7.9]).

The following proposition relates minimal left ideals in a ring R to simple left R-modules. Recall that a left R-module M is *simple* (or *irreducible*) if $M \neq \{0\}$ and M has no proper nonzero submodules.

Proposition 1 (Rotman [26])**.**

(i) Every minimal left ideal L in a ring R is a simple left R-module.
(ii) If R is left artinian, then every nonzero left ideal I contains a minimal left ideal.

Thus, the above proposition applies to Clifford algebras $C\ell_{p,q}$: every left spinor ideal S in $C\ell_{p,q}$ is a simple left $C\ell_{p,q}$

Thus, Proposition 2 tells us that every finitely generated left (or right) vector space V over a division ring D has a left (a right) dimension, which may be denoted $\dim V$. In [16] Jacobson gives an example of a division ring D and an abelian group V, which is both a right and a left D-vector space, such that the left and the right dimensions are not equal. In our discussion, spinor minimal ideals S will always be a left $C\ell_{p,q}$-module and a right K-module.

Since semisimple rings generalize the concept of a group algebra CG for a finite group G (cf. [17, 26]), we first discuss semisimple modules over a ring R.

Definition 3. A left R-module is *semisimple* if it is a direct sum of (possibly infinitely many) simple modules.

The following result is an important characterization of semisimple modules.

Proposition 3 (Rotman [26]). *A left R-module M over a ring R is semisimple if and only if every submodule of M is a direct summand.*

Recall that if a ring R is viewed as a left R-module, then its submodules are its left ideals, and, a left ideal is minimal if and only if it is a simple left R-module [26].

Definition 4. A ring R is *left semisimple* [3] if it is a direct sum of minimal left ideals.

One of the important consequences of the above for the theory of Clifford algebras, is the following proposition.

Proposition 4 (Rotman [26]). *Let R be a left semisimple ring.*

(i) *R is a direct sum of finitely many minimal left ideals.*
(ii) *R has both chain conditions on left ideals.*

From a proof of the above proposition one learns that, while $R = \bigoplus_i L_i$, that is, R is a direct sum of finitely-many left minimal ideals, the unity 1 decomposes into a sum $1 = \sum_i f_i$ of mutually annihilating primitive idempotents f_i, that is, (f_i)

Lemma 1 (Rotman [26]). Let R be a semisimple ring, and let

$$R = L_1 \quad L_n = B_1 \quad B_m \tag{4}$$

where the L_j are minimal left ideals and the B_i are the corresponding simple components of R.

- (i) Each B_i is a ring that is also a two-sided ideal in R, and $B_i B_j = (0)$ if $i \neq j$:
- (ii) If L is any minimal left ideal in R, not necessarily occurring in the given decomposition of R, then $L \cong L_i$ for some i and $L \quad B_i$:
- (iii) Every two-sided ideal in R is a direct sum of simple components.
- (iv) Each B_i is a simple ring.

Thus, we will gather from the Structure Theorem, that for simple Clifford algebras $C\ell_{p;q}$ we have only one simple component, hence $m = 1$, and thus all 2^k left minimal ideals generated by a complete set of 2^k primitive mutually annihilating idempotents which provide an orthogonal decomposition of the unity 1 in $C\ell_{p;q}$ (see part (c) of the theorem and notation therein). Then, for semisimple Clifford algebras $C\ell_{p;q}$ we have obviously $m = 2$.

Furthermore, we have the following corollary results.

Corollary 3 (Rotman [26]).

- (1) The simple components $B_1; \ldots; B_m$ of a semisimple ring R do not depend on a decomposition of R as a direct sum of minimal left ideals;
- (2) Let A be a simple artinian ring. Then,
  - (i) $A = \mathrm{Mat}(n; D)$ for some division ring D. If L is a minimal left ideal in A, then every simple left A-module is isomorphic to L; moreover, $D^{op} = \mathrm{End}_A(L)$:[5]
  - (ii) Two finitely generated left A-modules M and N are isomorphic if and only if $\dim_D(M) = \dim_D(N)$:

As we can see, part (1) of this last corollary gives a certain invariance in the decomposition of a semisimple ring into a direct sum of simple components. Part (2i), for the left artinian Clifford algebras $C\ell_{p;q}$ implies that simple Clifford algebras ($p - q \neq 1 \bmod 4$) are simple algebras .97535(n065(l)-9.06 0[(m)-3(o)4498(2)7.77624())s)-316.1-0.64204d [(6(f)806463(n)22.0293(t)

Thus, the above results, and especially the Wedderburn-Artin Theorem (parts I and II), shed a new light on the main Structure Theorem given in the following section. In particular, we see it as a special case of the theory of semisimple rings, including the left artinian rings, applied to the finite dimensional Clifford algebras $C\ell_{p,q}$.

$\hat{K} = \{ \hat{\ } \mid \ \in K$

is a decomposition of the Clifford algebra $C\ell_{p,q}$ into a direct sum of minimal left ideals, or, simple left $C\ell_{p,q}$-modules.

Part (d) determines the unique division ring $K = fC\ell_{p,q}f$, where f is any primitive idempotent, prescribed by the Wedderburn-Artin Theorem, such that the decomposition (9) or (11) is valid, depending whether the algebra is simple or not. This part also reminds us that the left spinor ideals, while remaining left $C\ell_{p,q}$ modules, are right K-modules. This is important when computing actual matrices in spinor representations (faithful and irreducible). Detailed computations of these representations in both simple and semisimple cases are shown in [2]. Furthermore, package CLIFFORD has a built-in database which displays matrices representing generators of $C\ell_{p,q}$, namely $e_1, \dots, e_n$, $n = p + q$, for a certain choice of a primitive idempotent f. Then, the matrix representing any element $u \in C\ell_{p,q}$ can the be found using the fact that the maps shown on Parts (e) and (f), are algebra maps.

Finally, we should remark, that while for simple Clifford algebras the spinor minimal left ideal carries a *faithful* (and irreducible) representation, that is, $\ker = \{1\}$; in the case of semisimple algebras, each $\frac{1}{2}$ spinor space S and $\hat{S}$ carries an irreducible but not faithful representation. Only in the double spinor space $S \oplus \hat{S}$; one can realize the semisimple algebra faithfully. For all practical purposes, this means that each element u in a semisimple algebra must be represented by a pair of matrices, according to the isomorphism (11). In practice, the two matrices can then be considered as a single matrix, but over $K \oplus \hat{K}$ which is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$, depending whether $p - q = 1 \bmod 8$; or $p - q = 5 \bmod 8$: We have already remarked earlier that while K is a division ring, $K \oplus \hat{K}$ is not.

4.