DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

ON CLIFFORD ALGEBRAS AND RELATED FINITE GROUPS AND GROUP ALGEBRAS

R. ABŁAMOWICZ

October 2016

No. 2016-2

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

ON CL O D AL B A AND LA D N O AND O AL B A

A o cz

Department of Mathematics, Tennessee Technological University
Cookeville, TN 38505, U.S.A.
rab amo cz@ n ec .edu, p://ma . n ec .edu/rafa /

A C Albuquerque and Majid [8] have shown how to view Cliord algebras $C\ell_{p,q}$ **as twisted group rings whereas Chernov has observed [13] that C**

1. N C N

The main goal of this survey paper is to show how certain finite groups, in particular, Salingaros vee groups [29–31], and elementary abelian group ($\mathbb{Z}_2$)7624(r)-0.6444981[(2)-(a)738u7□03-13.92□7

that the "transposition" anti-involution of $C\ell_{p,q}$ introduced in [3–5] is actually the antipode in the Hopf algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$.[1]

Our standard references on the group theory are [15, 17, 27]; in particular, for the theory of p-groups we rely on [24]; for Clifford algebras we use [14, 20, 21] and references therein; on representation theory we refer to [19]; and for the theory of Hopf algebras we refer to [25].

2. Clifford Algebras as Images of Group Algebras

Using Chernov's idea [13], in this section we want to show how Clifford algebras $C\ell_{p,q}$ can be viewed as images of group algebras $\mathbb{R}[G]$ of certain 2-groups. It is conjectured [34] that the group G, up to an isomorphism, is the Salingaros vee group $G_{p,q}$ [29–31]. These groups, and their subgroups, have been recently discussed in [4, 5, 11, 22, 23].

Definition 1. Let G be a finite group and let F be a field[2]. Then the group algebra $F[G]$ is the vector space

$$F[G] = \Big\{ \sum_{g \in G} \alpha_g g, \ \alpha_g \in F \Big\} \tag{1}$$

with multiplication defined as

$$\Big(\sum_{g \in G} \alpha_g g\Big)\Big(\sum_{h \in G} \beta_h h\Big) = \sum_{g,h \in G} \alpha_g \beta_h (gh) = \sum_{g \in G}\sum_{h \in G} \alpha_h \beta_{h^{-1}g}\, g \tag{2}$$

where all $\alpha_g, \beta_h \in F$. [19]

Thus, group algebras are associative unital algebras with the group identity element playing the role of the algebra identity. In the theory of representations of finite groups, all irreducible inequivalent representations are related to a complete decomposition of the group algebra over $\mathbb{C}$ viewed as a regular $\mathbb{C}$-module (cf. [19, Maschke Theorem]). The theory is rich on its own. The theory of group characters can then be derived from the representation theory [19], or, as it is often done, from the combinatorial arguments and the theory of characters of the symmetric group [28]. Since in this survey we are only interested in finite groups, we just recall for completeness that every finite group is isomorphic to a subgroup of a symmetric group [27].

We begin by recalling a definition of a p-group.

Definition 2. Let p be a prime. A group G is a p-group if every element in G is of order p^k for some $k \geq 1$.

Note that any finite group G of order p^n is a p-group. A classical result states that a center of any p-group is nontrivial, and, by Cauchy's theorem we know that every finite p-group has an element of order p. Thus, in particular, the center of any finite p-group has an element of order p [15, 17, 27]. In the following, we will be working only with finite 2-groups such as, for example, the group $(\mathbb{Z}_2)^n$ and Salingaros vee groups $G_{p,q}$ of order 2^{1+p+q}.

---

[1] We remark that twisted group rings can also be described as certain special Ore extensions known as skew polynomial rings [12].

[2] Usually, $F = \mathbb{R}$ or $\mathbb{C}$ although finite fields are also allowed. In this paper, we will be looking at the real Clifford algebras $C\ell_{p,q}$ as images of real group algebras or as real twisted group algebras.

Two important groups in the theory of finite 2-groups and hence in this paper, are the quaternionic group Q_8 and the dihedral group D_8 (the symmetry group of a square under rotations and reflections), both of order $|Q_8| = |D_8| = 8$. These groups have the following presentations:

Definition 3. The quaternionic group Q_8 has the following two presentations:

(3a) $$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$$

(3b) $$= \langle \tau, I, J \mid \tau^2 = 1, I^2 = J^2 = \tau, IJ = \tau JI \rangle$$

Thus, $Q_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where the group elements have orders as follows: $|a^2| = 2$, $|a| = |a^3| = |b| = |ab| = |a^2b| = |a^3b| = 4$, so the order structure of Q_8 is $(1, 1, 6)$;[3] and the center $Z(Q_8) = \{1, a^2\} \cong \mathbb{Z}_2$. Here, we can choose $\tau = a^2$. While the presentation (3a) uses only two generators, for convenience and future use, we prefer presentation (3b) which explicitly uses a central element τ of order 2.

Definition 4. The dihedral group D_8 (the symmetry group of a square) has the following two presentations:

(4a) $$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$$

(4b) $$= \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$$

Thus, $D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $|a^2| = |b| = |ab| = |a^2b| = |a^3b| = 2$, $|a| = |a^3| = 4$, the order structure of D_8 is $(1, 5, 2)$, and $Z(D_8) = \{1, a^2\} \cong \mathbb{Z}_2$. Here, we can choose $\tau = b$, $\sigma = a$, hence, $\sigma^2 \in Z(D_8)$. That is, σ^2 is our central element of order 2, and our preferred presentation of D_8 is (4b).

In the following two examples, we show how one can construct the Clifford algebra $C\ell_{0,2} \cong \mathbb{H}$ (resp. $C\ell_{1,1}$) as an image of the group algebra of Q_8 (resp. D_8).

Example 1. (Constructing $\mathbb{H} \cong C\ell_{0,2}$ as $\mathbb{R}[Q_8]/\mathcal{J}$)
Define an algebra map ψ from the group algebra $\mathbb{R}[Q_8] \to \mathbb{H} = \operatorname{span}_{\mathbb{R}}\{1, i, j, ij\}$ as follows:

(5) $$1 \mapsto 1, \quad \tau \mapsto -1, \quad I \mapsto i, \quad J \mapsto j,$$

Then, $\mathcal{J} = \ker \psi = (1 + \tau)$ for the central element τ of order 2 in Q_8[4], so $\dim_{\mathbb{R}} \mathcal{J} = 4$ and ψ is surjective. Let $\pi : \mathbb{R}[Q_8] \to \mathbb{R}[Q_8]/\mathcal{J}$ be the natural map $u \mapsto u + \mathcal{J}$. There exists an isomorphism $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to \mathbb{H}$ such that $\varphi \circ \pi = \psi$ and

$$\pi(I^2) = I^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(I^2)) = \psi(\tau) = -1 = (\psi(I))^2 = i^2;$$
$$\pi(J^2) = J^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(J^2)) = \psi(\tau) = -1 = (\psi(J))^2 = j^2;$$
$$\pi(IJ + JI) = IJ + JI + \mathcal{J} = (1 + \tau)JI + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(IJ + JI)) = \psi(0) = 0 = \psi(I)\psi(J) + \psi(J)\psi(I) = ij + ji.$$

Thus, $\mathbb{R}[Q_8]/\mathcal{J} \cong \psi(\mathbb{R}[Q_8]) = \mathbb{H} \cong C\ell_{0,2}$ provided the central element τ is mapped to -1 (see also [13]).

Example 2. (Constructing $C`_{1;1}$ as $R[D_8]=J$)

Theorem 1 (Chernov). *Let G be a finite 2-group of order 2^{1+n} generated by a central element τ of order 2 and additional elements $g_1, \dots, g_n$, which satisfy the following relations:*

$$\tau^2 = 1, \quad (g_1)^2 = \cdots = (g_p)^2 = 1, \quad (g_{p+1})^2 = \cdots = (g_{p+q})^2 = \tau; \tag{8a}$$

$$\tau g_j = g_j \tau, \quad g_i g_j = \tau g_j g_i, \quad i, j = 1, \dots, n = p + q; \tag{8b}$$

so that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \mid \alpha_k \in \{0, 1\}, k = 0, 1, \dots, n\}$. Let $J = (1 + \tau)$ be an ideal in the group algebra $\mathbb{R}[G]$ and let $C\ell_{p,q}$ be the universal real Clifford algebra generated by $\{e_k\}, k = 1, \dots, n = p + q$, where

$$e_i^2 = Q(e_i) \cdot 1 = \varepsilon_i \cdot 1 = \begin{cases} 1 & \text{for } 1 \le i \le p; \\ -1 & \text{for } p+1 \le i \le p+q; \end{cases} \tag{9a}$$

$$e_i e_j + e_j e_i = 0, \quad i \ne j, \quad 1 \le i, j \le n. \tag{9b}$$

Then, (a) $\dim_{\mathbb{R}} J = 2^n$; (b) There exists a surjective algebra homomorphism ψ from the group algebra $\mathbb{R}[G]$ to $C\ell_{p,q}$ so that $\ker \psi = J$ and $\mathbb{R}[G]/J \cong C\ell_{p,q}$.

Remark 1. Chernov's theorem does not give the existence of the group G. It only states that should such group exist whose generators satisfy relations (8), the result follows. It is not difficult to conjecture that the group G in that theorem is in fact the Salingaros vee group $G_{p,q}$, that is, $\mathbb{R}[G_{p,q}]/J \cong C\ell_{p,q}$ (see [34]). In fact, we have seen it in Examples 1 and 2 above.

Chernov's theorem. Observe that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} \mid \alpha_k \in \{0, 1\}, k = 0, 1, \dots, n\}$. The existence of a central element of order 2 is guaranteed by a well-known fact that the center of any p-group is nontrivial, and by Cauchy Theorem. [27] Define an algebra homomorphism $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ such that

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad g_j \mapsto e_j, \quad j = 1, \dots, n. \tag{10}$$

Clearly, $J \subseteq \ker \psi$. Let $u \in \mathbb{R}[G]$. Then,

$$u = \sum_{\alpha} \lambda_\alpha \tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} = u_1 + \tau u_2 \tag{11}$$

where

$$u_i = \sum_{e} \lambda_e^{(i)} g_1^{\alpha_1} \cdots g_n^{\alpha_n}, \quad i = 1, 2; \tag{12a}$$

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} \quad \text{and} \quad e = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n. \tag{12b}$$

Then, since $(g_j$

and a $\mathbb{K}$-valued, where $\mathbb{K} = f C\ell_{p,q} f$, spinor norm $(\psi, \phi) = T_{\varepsilon}(\tilde{\psi}\phi)$ on S invariant under (infinite) group $G^{\varepsilon}_{p,q}$ (with $G_{p,q} < G^{\varepsilon}_{p,q}$) different, in general, from spinor norms related to reversion and conjugation in $C\ell_{p,q}$.

$G_{p,q}$ act transitively on a complete set $\mathcal{F}$, $|\mathcal{F}| = 2^{q - r_{q-p}}$, of mutually annihilating primitive idempotents where r_i is the Radon-Hurwitz number. See a footnote in Appendix A for a definition of r_i.

The normal stabilizer subgroup $G_{p,q}(f) \lhd G_{p,q}$ of f is of order $2^{1+p+r_{q-p}}$ and monomials m_i in its (non-canonical) left transversal together with f determine a spinor basis in S.

The stabilizer groups $G_{p,q}(f)$ and the invariance groups $G^{\varepsilon}_{p,q}$ of the spinor norm have been classified according to the signature (p,q) for $(p+q) \le 9$ in simple and semisimple algebras $C\ell_{p,q}$.

$G_{p,q}$ permutes the spinor basis elements modulo the commutator subgroup $G'_{p,q}$ by left multiplication.

The ring $\mathbb{K} = f C\ell_{p,q} f$ is $G_{p,q}$-invariant.

3.2. Important Finite Subgroups of $C\ell_{p,q}$. In this section, we summarize properties and definitions of some finite subgroups of the group of invertible elements $C\ell^{*}_{p,q}$ in the Clifford algebra $C\ell_{p,q}$. These groups were defined in [3–5].

$G_{p,q}$ – Salingaros vee group of order $|G_{p,q}| = 2^{1+p+q}$,

$G'_{p,q} = \{\pm 1\}$ – the commutator subgroup of $G_{p,q}$,

Let $O(f)$ be the orbit of f under the conjugate action of $G_{p,q}$, and let $G_{p,q}(f)$ be the stabilizer of f. Let

$$N = |\mathcal{F}| = [G_{p,q} : G_{p,q}(f)] = |O(f)| = |G_{p,q}|/|G_{p,q}(f)| = 2 \cdot 2^{p+q}/|G_{p,q}(f)| \tag{25}$$

then $N = 2^k$ (resp. $N = 2^{k-1}$) for simple (resp. semisimple) $C\ell_{p,q}$ where $k = q - r_{q-p}$ and $[G_{p,q} : G_{p,q}(f)]$ is the index of $G_{p,q}(f)$ in $G_{p,q}$.

$G_{p,q}(f) \lhd G_{p,q}$ and $|G_{p,q}(f)| = 2^{1+p+r_{q-p}}$ (resp. $|G_{p,q}(f)| = 2^{2+p+r_{q-p}}$) for simple (resp. semisimple) $C\ell_{p,q}$.

The set of commuting monomials $\mathcal{T} = \{e_{\underline{i}_1}, \ldots, e_{\underline{i}_k}\}$ (squaring to 1) in the primitive idempotent $f = \frac{1}{2}(1 \pm e_{\underline{i}_1}) \cdots \frac{1}{2}(1 \pm e_{\underline{i}_k})$ is point-wise stabilized by $G_{p,q}(f)$.

$T_{p,q}(f) := \langle -1, \mathcal{T} \rangle = G'_{p,q} \times \langle e_{\underline{i}_1}, \ldots, e_{\underline{i}_k} \rangle \cong G'_{p,q} \times (\mathbb{Z}_2)^k$; the idempotent group of f with $|T_{p,q}(f)| = 2^{1+k}$,

$K_{p,q}(f) = \langle -1, m \mid m \in \mathcal{K} \rangle < G_{p,q}(f)$ – the field group of where f is a primitive idempotent in $C\ell_{p,q}$, $\mathbb{K} = f C\ell_{p,q} f$, and $\mathcal{K}$ is a set of monomials (a transversal) in $\mathcal{B}$ which span $\mathbb{K}$ as a real algebra. Thus,

$$|K_{p,q}(f)| = \begin{cases} 2, & p - q = 0, 1, 2 \bmod 8; \\ 4, & p - q = 3, 7 \bmod 8; \\ 8, & p - q = 4, 5, 6 \bmod 8 \end{cases} \tag{26}$$

$G^{\varepsilon}_{p,q} = \{ g \in C\ell_{p,q} \mid T_{\varepsilon}(\tilde{g} g) = 1 \}$ (infinite group)

Before we state the main theorem from [5] that relates the above finite groups to the Salingaros vee groups, we recall the definition of a transversal.

Definition 7. Let K be a subgroup of a group G. A transversal ℓ of K in G is a subset of G consisting of exactly one element $\ell(bK)$ from every (left) coset bK, and with $\ell(K) = 1$.

Theorem 2 (Main Theorem). *Let f be a primitive idempotent in $C\ell_{p,q}$ and let $G_{p,q}$, $G_{p,q}(f)$, $T_{p,q}(f)$, $K_{p,q}(f)$, and $G^0_{p,q}$ be the groups defined above. Let $S = C\ell_{p,q}f$ and $\mathbb{K} = fC\ell_{p,q}f$.*

(i) Elements of $T_{p,q}(f)$ and $K_{p,q}(f)$ commute.
(ii) $T_{p,q}(f) \cap K_{p,q}(f) = G^0_{p,q} = \{\pm 1\}$.
(iii) $G_{p,q}(f) = T_{p,q}(f)K_{p,q}(f) = K_{p,q}(f)T_{p,q}(f)$.
(iv) $|G_{p,q}(f)| = |T_{p,q}(f)K_{p,q}(f)| = \frac{1}{2}|T_{p,q}(f)||K_{p,q}(f)|$.
(v) $G_{p,q}(f) \lhd G_{p,q}$, $T_{p,q}(f) \lhd G_{p,q}$, and $K_{p,q}(f) \lhd G_{p,q}$. In particular, $T_{p,q}(f)$ and $K_{p,q}(f)$ are normal subgroups of $G_{p,q}(f)$.
(vi) We have:

$$G_{p,q}(f)/K_{p,q}(f) \cong T_{p,q}(f)/G^0_{p,q}, \tag{27}$$

$$G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/G^0_{p,q}. \tag{28}$$

(vii) We have:

$$(G_{p,q}(f)/G^0_{p,q})/(T_{p,q}(f)/G^0_{p,q}) \cong G_{p,q}(f)/T_{p,q}(f) \cong K_{p,q}(f)/\{\pm 1\} \tag{29}$$

and the transversal of $T_{p,q}(f)$ in $G_{p,q}(f)$ spans $\mathbb{K}$ over $\mathbb{R}$ modulo f.
(viii) The transversal of $G_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{K}$ modulo f.
(ix) We have $(G_{p,q}(f)/T_{p,q}(f)) \lhd (G_{p,q}/T_{p,q}(f))$ and

$$(G_{p,q}/T_{p,q}(f))/(G_{p,q}(f)/T_{p,q}(f)) \cong G_{p,q}/G_{p,q}(f) \tag{30}$$

and the transversal of $T_{p,q}(f)$ in $G_{p,q}$ spans S over $\mathbb{R}$ modulo f.
(x) The stabilizer $G_{p,q}(f)$ can be viewed as

$$G_{p,q}(f) = \bigcap_{x \in T_{p,q}(f)} C_{G_{p,q}}(x) = C_{G_{p,q}}(T_{p,q}(f)) \tag{31}$$

where $C_{G_{p,q}}(x)$ is the centralizer of x in $G_{p,q}$ and $C_{G_{p,q}}(T_{p,q}(f))$ is the centralizer of $T_{p,q}(f)$ in $G_{p,q}$.

3.3. Summary of Some Basic Properties of Salingaros Vee Groups

Theorem 3. Let $G_{p,q} \subset C\ell_{p,q}$. Then,

$$
(32) \qquad Z(G_{p,q}) = \begin{cases} \{\pm 1\} \cong \mathbb{Z}_2 & \text{if } p - q \equiv 0, 2, 4, 6 \pmod 8; \\ \{\pm 1, \pm \beta\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p - q \equiv 1, 5 \pmod 8; \\ \{\pm 1, \pm \beta\} \cong \mathbb{Z}_4 & \text{if } p - q \equiv 3, 7 \pmod 8. \end{cases}
$$

as a consequence of $Z(C\ell_{p,q}) = \{1\}$ (resp. $\{1, \beta\}$) when $p+q$ is even (resp. odd) where $\beta = e_1 e_2 \cdots e_n$, $n = p+q$, is the unit pseudoscalar in $C\ell_{p,q}$.
In Salingaros' notation, the five isomorphism classes denoted as $N_{2k-1}, N_{2k}, \Omega_{2k-1}, \Omega_{2k}, S_k$ correspond to our notation $G_{p,q}$ as follows:

Definition 9 (Dornhoff [15]). A finite p-group P is extra-special if (i) $P' = Z(P)$; (ii) $|P'| = p$; and (iii) P/P' is elementary abelian.

Example 4. (D_8 is extra-special)
$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ is extra-special because:

$$Z(D_8) = D_8' = [D_8, D_8] = \langle a^2 \rangle,\ |Z(D_8)| = 2;$$
$$D_8/D_8' = D_8/Z(D_8) = \langle \langle a^2 \rangle, a\langle a^2 \rangle, b\langle a^2 \rangle, ab\langle a^2 \rangle \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Example 5. (Q_8 is extra-special)
$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ is extra-special because:

$$Z(Q_8) = Q_8' = [Q_8, Q_8] = \langle a^2 \rangle,\ |Z(Q_8)| = 2;$$
$$Q_8/Q_8' = Q_8/Z(Q_8) = \langle \langle a^2 \rangle, a\langle a^2 \rangle, b\langle a^2 \rangle, ab\langle a^2 \rangle \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Let us recall now definitions of internal and external central products of groups.

Definition 10 (Gorenstein [17]).

(1) A group G is an *internal central product* of two subgroups H and K if:
  (a) $[H, K] = \langle 1 \rangle$;
  (b) $G = HK$;
(2) A group G is an *external central product* $H \circ K$ of two groups H and K with $H_1 \subseteq Z(H)$ and $K_1 \subseteq Z(K)$ if there exists an isomorphism $\theta : H_1 \to K_1$ such that G is $(H \times K)/N$ where
$$N = \{(h, \theta(h^{-1})) \mid h \in H_1\}.$$
Clearly: $N \triangleleft (H \times K)$ and $|H \circ K| = |H||K|/|N| \leq |H \times K| = |H||K|$.

Here we recall an important result on extra-special p-groups as central products.

Lemma 1 (Leedham-Green and McKay [24]) *An extra-special p-group has order p^{2n+1} for some positive integer n, and is the iterated central product of non-abelian groups of order p^3.*

As a consequence, we have the following lemma and a corollary. For their proofs, see [11].

Lemma 2. $Q_8 \circ Q_8 = D_8 \circ D_8 \ncong D_8 \circ Q_8$, where D_8 is the dihedral group of order 8 and Q_8 is the quaternion group of order 8 [illegible]

2: $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$.

where it is understood that these are iterated central products; that is, $D_8 \circ D_8 \circ D_8$ is really $(D_8 \circ D_8) \circ D_8$ and so on.

Thus, the above theorem now explains the following theorem due to Salingaros regarding the iterative central product structure of the finite 2-groups named after him.

Theorem 5 (Salingaros Theorem [31]) Let $N_1 = D_8$, $N_2 = Q_8$, and $(G)^{\circ k}$ be the iterative central product $G \circ G \circ \cdots \circ G$ (k times) of G. Then, for $k \geq 1$:

(1) $N_{2k-1} = (N_1)^{\circ k} = (D_8)^{\circ k}$,
(2) $N_{2k} = (N_1)^{\circ k} \circ N_2 = (D_8)^{\circ (k-1)} \circ Q_8$,
(3) $\Omega_{2k-1} = N_{2k-1} \circ (Z_2 \times Z_2) = (D_8)^{\circ k} \circ (Z_2 \times Z_2)$,
(4) $\Omega_{2k} = N_{2k} \circ (Z_2 \times Z_2) = (D_8)^{\circ (k-1)} \circ Q_8 \circ (Z_2 \times Z_2)$,
(5) $S_k = N_{2k-1} \circ Z_4 = N_{2k} \circ Z_4 = (D_8)^{\circ k} \circ Z_4 = (D_8)^{\circ (k-1)} \circ Q_8 \circ Z_4$.

In the above theorem:

- Z_2, Z_4 are cyclic groups of order 2 and 4, respectively;
- D_8 and Q_8 are the dihedral group of a square and the quaternionic group;
- $Z_2 \times Z_2$ is elementary abelian of order 4;
- N_{2k-1} and N_{2k} are extra-special of order 2^{2k+1}; e.g., $N_1 = D_8$ and $N_2 = Q_8$;
- $\Omega_{2k-1}, \Omega_{2k}, S_k$ are of order 2^{2k+2}.
- $\circ$ denotes the iterative central product of groups with, e.g., $(D_8)^{\circ k}$ denotes the iterative central product of k-copies of D_8, etc.,

We can tabulate the above results for Salingaros vee groups $G_{p,q}$ of orders ≤ 256, $(p+q \leq 7)$ (Brown [11]) in the following table:

Table 2. Salingaros Vee Groups $|G_{p,q}| \leq 256$

Isomorphism Class	Salingaros Vee Groups
N_{2k}	$N_0 = G_{0,0}$; $N_2 = Q_8 = G_{0,2}$; $N_4 = G_{4,0}$; $N_6 = G_{6,0}$
N_{2k-1}	$N_1 = D_8 = G_{2,0}$; $N_3 = G_{3,1}$; $N_5 = G_{0,6}$
Ω_{2k}	$\Omega_0 = G_{1,0}$; $\Omega_2 = G_{0,3}$; $\Omega_4 = G_{5,0}$; $\Omega_6 = G_{6,1}$
Ω_{2k-1}	$\Omega_1 = G_{2,1}$; $\Omega_3 = G_{3,2}$; $\Omega_5 = G_{0,7}$
S_k	$S_0 = G_{0,1}$; $S_1 = G_{3,0}$; $S_2 = G_{4,1}$; $S_3 = G_{7,0}$

5. Clifford Algebras Modeled with Walsh Functions

Until now, the finite 2-groups such as the Salingaros vee groups $G_{p,q}$ have appeared either as finite subgroups of the group of units $C\ell^{\times}_{p,q}$ in the Clifford algebra, or, as groups whose group algebra modulo a certain ideal generated by $1+\tau$ for some central element τ of order 2 was isomorphic to the given Clifford algebra $C\ell_{p,q}$. In these last two sections, we recall how the (elementary abelian) group $(\mathbb{Z}_2)^n$ can be used to define a Clifford product on a suitable vector space.

In this section, we recall the well-known construction of the Clifford product on the set of monomial terms e_a

Remark 2. Observe that if the scalar factor in front of $e_{\underline{a}\ \underline{b}}$ in (37) were set to be identically equal to 1, then we would have $e_{\underline{a}}e_{\underline{b}} = e_{\underline{b}}e_{\underline{a}}$ for any $e_{\underline{a}}, e_{\underline{b}} \in A$. Thus, the algebra A would be isomorphic to the (abelian) group algebra $\mathbb{R}[G]$ where $G = (\mathbb{Z}_2)^n$. That is, the scalar factor introduces a twist in the algebra product in A and so it makes A, hence the Clifford algebra $C\ell_{p,q}$, isomorphic to the twisted group algebra $\mathbb{R}^t[(\mathbb{Z}_2)^n]$.

Formula (37) is encoded as a procedure cmulWalsh3 in CLIFFORD, a Maple package for computations with Clifford algebras [2, 7]. It has the following pseudo-code.

```
cmulWalsh3:=proc(eI::clibasmon,eJ::clibasmon,B1::{matrix,list(nonnegint)})
local
```

7. Conclusions

As stated in the Introduction, the main goal of this survey pa

Furthermore, $C\ell_{p,q}$ has a complete set of 2^k such primitive mutually annihilating idempotents which add up to the unity 1 of $C\ell_{p,q}$.

(d) When $(p \quad q) \bmod 8$ is $0, 1, 2,$ or $3, 7,$ or $4, 5, 6$, then the division ring $K = f C\ell_{p,q} f$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, and the map $S \quad K \to S; (\; ; \;) \mapsto$ defines a right K-module structure on the minimal left ideal $S = C\ell_{p,q} f$.

(e) When $C\ell_{p,q}$ is simple, then the map

$$C\ell_{p,q} \to \mathrm{End}_K(S); \quad u \mapsto (u); \quad (u) \quad = u \tag{41}$$

gives an irreducible and faithful representation of $C\ell_{p,q}$ in S.

(f) When $C\ell_{p,q}$ is semisimple, then the map

$$C\ell_{p,q} \to \mathrm{End}_{K \quad \hat{K}}(S \quad \hat{S}); \quad u \mapsto (u); \quad (u) \quad = u \tag{42}$$

gives a faithful but reducible representation of $C\ell_{p,q}$ in the double spinor space $S \quad \hat{S}$ where $S = \{uf \mid u \in C\ell_{p,q}\}$, $\hat{S} = \{u\hat{f} \mid u \in C\ell_{p,q}\}$ and $\hat{}$ stands for the grade-involution in $C\ell_{p,q}$. In this case, the ideal $S \quad \hat{S}$

[16] **H. B. Downs:** Cli ord Algebras as Hopf Algebras and the Connection Betwe Cocycles and Walsh Functions**, Master Thesis (in progress), Department of Mathematics, TTU, Cookeville, TN (May 2017, expected).**

[17] **D. Gorenstein,**