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REPRESENTATIONS AND CHARACTERS OF SALINGAROS' VEE GROUPS OF LOW ORDER

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Proposition 1. Let V be a G-module,

the conjugacy classes of the dihedral group D_{2n}. [12]

Example 2. Consider the dihedral group $D_{2n} = \langle r, p : r^n = p^2 = (rp)^2 = e \rangle$ of order $2n$. When n is odd, D_{2n} has [1]

Proposition 3. Let G have k conjugacy classes with representatives $g_1, g_2, \ldots, g_k$. Also, let χ and ψ be some characters of G. Then, $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$, and

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) = \sum_{i=1}^{k} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|} \tag{7}$$

Theorem 4. Let χ and ψ be irreducible characters of a group G. The characters are orthonormal with respect to the inner product, i.e., $\langle \chi, \psi \rangle = \delta_{\chi,\psi}$.

As a consequence of the above theorem, several results can be stated in relation to representations, irreducibility, etc. The following theorem will be used extensively in finding irreducible characters of certain groups in Section 2.

Example 3. Let $G = S_n$: Of course it is well known that the number of conjugacy classes for any S_n equals the number of partitions of n: Furthermore, each class consists of permutations having the same cycle structure because the action of conjugation preserves the cycle structure.

Example 4. Let $G = D_{2n}$: The number of conjugacy classes for the dihedral group was discussed in Example 2. For example, D_6 and D_8

2.1. General definitions and properties

The vee groups were introduced by Salingaros in [18–20]. They were more recently studied in [2–4, 22] where they were denoted as $G_{p,q}$. In particular, these groups are central extensions of extra-special 2-groups. [6, 9–11, 13, 22]

Definition 3. Let $C\ell_{p,q}$ be the real Clifford algebra of a non-degenerate quadratic form with signature (p,q) and let $B = \{e_{\underline{i}} \mid 0 \le |\underline{i}| \le n\}$ be a basis for $C\ell_{p,q}$ consisting of basis monomials $e_{\underline{i}} = e_{i_1}e_{i_2}\cdots e_{i_k}$, $i_1 < i_2 < \cdots < i_k$, for $0 \le k \le n$ where $n = p+q$.

where $\beta = e_1 e_2 \cdots e_n$, $n = p+q$, is the unit pseudoscalar in $C\ell_{p,q}$. This leads to the following conclusion (see also [22]).

Theorem 8. Let $G_{p,q} \subset C\ell_{p,q}$. Then,

$$(13) \qquad Z(G_{p,q}) = \begin{cases} \{\pm 1\} = \mathbb{Z}_2 & \text{if } p-q \equiv 0,2,4,6 \pmod 8; \\ \{\pm 1, \pm\beta\} = \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p-q \equiv 1,5 \pmod 8; \\ \{\pm 1, \pm\beta\} = \mathbb{Z}_4 & \text{if } p-q \equiv 3,7 \pmod 8. \end{cases}$$

The following result implies that the vee groups of order 2^2 are abelian. For the proof of this proposition, see [16].

Proposition 5. If p is a prime, then every group G of order p^2 is abelian.

It is worth to know the order relation of the normal subgroups of the Salingaros' vee groups.

Proposition 6. If a group G is of order $|G| = p^n$, then G has a normal subgroup of order p^m for every $m \leq n$.

So, this result tells that $G_{p,q}$ of order 2^{p+q+1} has a normal subgroup of order 2^m for any $m \leq p+q+1$, which implies that $G_{p,q}$ are not simple groups.

2.2. Conjugacy classes

In this section we discuss the conjugacy classes of $G_{p,q}$ using Theorem 8. It is convenient to separately address the

Theorem 9. Let N be the number of conjugacy classes in $G_{p;q}$. Then,

$$N = \begin{cases} 1 + 2^{p+q} & \text{if } p+q \text{ is even;} \\ 2 + 2^{p+q} & \text{if } p+q \text{ is odd.} \end{cases} \tag{16}$$

Proof.

The following theorem gives the number of inequivalent representations of degree one of the group $G_{p,q}$.

Theorem 10. Let M be the number of inequivalent representations of degree one of $G_{p,q}$. Then,

$$M = \begin{cases} 2 \cdot 2^{p+q} = 4 & \text{if } p+q = 1; \\ 2^{p+q} & \text{if } p+q \geq 2. \end{cases} \tag{17}$$

Proof. From Theorem 2, the number of degree one representations of $G_{p,q}$ is the index of its commutator subgroup $[G_{p,q} : G'_{p,q}]$. When $p+q = 1$, the commutator subgroup $G'_{p,q} = \{1\}$ and so $M = [G_{p,q} : G'_{p,q}] = (2 \cdot 2^1)/1 = 4$. For $p+q \geq 2$, $G'_{p,q} = \{1, -1\}$, so $M = [G_{p,q} : G'_{p,q}] = (2 \cdot 2^{p+q})/2 = 2^{p+q}$, as desired. ∎

Note that Maschke's Theorem 1 gives the decomposition $\mathbb{C}[G_{p,q}] = \bigoplus_i^N m_i V^{(i)}$ and from Proposition 4, one gets $|\mathbb{C}[G_{p,q}]| = \sum_{i=1}^{N} m_i{}^2$. From the above theorem, provided that M is the number of degree one representations of the group, the dimension of the group algebra $\mathbb{C}[G_{p,q}]$ can be rewritten as

$$|\mathbb{C}[G_{p,q}]| = M + \sum_{i=M+1}^{N} m_i^2. \tag{18}$$

Thus, the difference $N - M$ is the number of inequivalent irreducible representations of $G_{p,q}$ with degree two or more. This can be formally stated as the following result.

Theorem 11. Let L be the number of inequivalent irreducible representations with degree two or more of $G_{p,q}$. (i) Let $p+q \geq 2$. If $p+q$ is even, then $L = 1$ otherwise $L = 2$. (ii) When $p+q = 1$, then $L = 0$.

Proof. The proof follows immediately from Theorems 9 and 10. ∎

In the remainder of this section, we give the order structure and conjugacy classes of Salingaros' vee groups of orders 4, 8, and 16.

Example 6. Consider the abelian groups $G_{1,0}$ and $G_{0,1}$. The number of conjugacy classes is $N = 2 + 2^1 = 4$ as predicted by Theorem 9, and the conjugacy classes are:

$$K_1 = \{1\}; \quad K_2 = \{-1\}; \quad K_3 = \{e_1\}; \quad K_4 = \{-e_1\}. \tag{19}$$

Since the groups $G_{1,0}$ and $G_{0,1}$ have the same conjugacy classes, what distinguishes them is their order structure. The order structure of these groups is summarized in Table 2 where C.O.S. and G.O.S. give the center order structure and the group order structure, respectively, of each group. Also, ${}^2\mathrm{Mat}(1;\mathbb{R})$ denotes $\mathrm{Mat}(1;\mathbb{R}) \oplus \mathrm{Mat}(1;\mathbb{R})$.

Tab. 2: Vee groups $G_{p;q}$ of order 4 for $p + q = 1$

(p,q)	Group	$C`_{p;q}$	Center	^{2}	C.O.S.	G.O.S.	L	M	N
(1; 0)	$G_{1;0} = D_4$	$^2\mathrm{Mat}(1;R)$	Z_2 Z_2	+1	(1; 3; 0)	(1; 3; 0)	0	4	4
(0; 1)	$G_{0;1} = Z_4$	$\mathrm{Mat}(1;C)$	Z_4	1	(1; 1; 2)	(1; 1; 2)	0	4	4

Example 7. Consider the non-abelian groupsle 7

Tab. 4: Vee groups $G_{p;q}$

One needs to find four vectors $u_1, u_2, u_3,$ and u_4 which span 1-dimensional $G_{1,0}$-invariant subspaces $V^{(1)}, V^{(2)}, V^{(3)},$ and $V^{(4)}$ such that

$$\mathbb{C}[G_{1,0}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)} \tag{22}$$

and $V^{(i)} = \operatorname{span}\{u_i\}, i = 1, \ldots, 4.$ Notice that all $V^{(i)}$ are of dimension 1 since the group is abelian and all irreducible modules are one dimensional. The following algorithm can be used to find the basis vectors.

Algorithm 1.

1: Let $G = S = G_{1,0}$ and $V = \mathbb{C}[S] = \mathbb{C}[G_{1,0}].$

2: Let u_1 be the sum of all basis elements in V and define $V^{(1)} = \operatorname{span}\{u_1\}$. Such subspace always carries the trivial representation and it is G-invariant since $gu_1 = u_1$ for every $g \in G$.

3: Compute a basis for the orthogonal complement of $V^{(1)}$ in V and rename this complement as V. This orthogonal complement is obviously 3-dimensional and it is G-invariant by Proposition 1.

4: Using Groebner basis technique [8], find a 1-dimensional G-invariant subspace $V^{(2)}$ in V and find its spanning vector $u_2.$

5: Find a 2-dimensional orthogonal complement of $V^{(2)}$ in V. Call this complement V. By the same reasoning, it is G-invariant.

6: Find a 1-dimensional G-invariant subspace $V^{(3)}$ in V different from $V^{(2)}$ and its spanning vector $u_3.$

7: Find a basis for the orthogonal complement $V^{(4)}$ of $V^{(1)} \oplus V^{(2)} \oplus V^{(3)}$ in $\mathbb{C}[G_{1,0}]$ and its spanning vector $u_4.$

8: The algorithm terminates since the dimension of $\mathbb{C}[G_{1,0}]$ is finite.

From the above procedure, one obtains all basis vectors u_i as linear combinations of the standard basis $B = \{1, -1, e_1, -e_1\}$ of $\mathbb{C}[G_{1,0}]$ as follows:

$$V^{(1)} = \operatorname{span}\{u_1\}, \quad u_1 = (1)1 + (1)(-1) + (1)(e_1) + (1)(-e_1);$$
$$V^{(2)} = \operatorname{span}\{u_2\}, \quad u_2$$

from the following character table.

(24)

char/class	K_1	K_2	K_3	K_4
$^{(1)}$	1	1	1	1
$^{(2)}$	1	1	1	1
$^{(3)}$	1	1	1	1
$^{(4)}$	1	1	1	1

The explicit matrix representations are shown in Table 12 in Appendix B. Note that in the character table, rows and columns are orthonormal. Let $^{(i)}$ denote the character of the representation $X^{(i)}$: So, for example, the inner product of the characters $^{(2)}$ and $^{(3)}$ from the above table is computed as follows:

$$\langle {}^{(2)}; {}^{(3)} \rangle = \frac{1}{4}\sum_{i=1}^{4} |K_i| \, {}^{(2)}_{K_i} \overline{{}^{(3)}_{K_i}} = \frac{1}{4}((1)(1) + (\quad 1)(\quad 1) + (\quad 1)(1) + (1)(\quad 1)) = 0$$

since $|K_i| = 1$ for each class. This verifies the character orthogonality relation of the from the character table.

(27)

char/class	K_1	K_2	K_3	K_4
(1)	1	1	1	1
(2)	1	1	i	i
(3)	1	1	1	1
(4)	1	1	i	i

2: Apply Algorithm 1 to find vectors $u_1; u_2; u_3; u_4$ providing bases for the one-dimensional G-invariant submodules $V^{(1)}; V^{(2)}; V^{(3)}; V^{(4)}$ in V.

3: Find a basis for the orthogonal complement of $V^{(1)}$ $V^{(2)}$ $V^{(3)}$ $V^{(4)}$ in V and call it V. It is 4-dimensional.

4: Using Groebner basis technique, find any 2-dimensional G-invariant subspace in V and call it $V^{(5)}$. That is, find its basis vectors u_5 and u_6:

5: Find a basis for the orthogonal complement of $V^{(5)}$ in V and call it $V^{(6)}$: That is, find its spanning vectors u_7 and u_8:

6: The algorithm terminates when all eight vectors $u_1; \ldots; u_8$ are found and these vectors provide a basis for the decomposition of $\mathbb{C}[G_{2;0}]$:

Once the decomposition of $\mathbb{C}[G_{2;0}]$ has been found, one can compute all irreducible representations $X^{(i)}; i = 1; \ldots; 6$; of $G_{2;0}$ in the six invariant submodules $V^{(i)}$: The degree-one representations $X^{(1)}; X^{(2)}; X^{(3)}$; and $X^{(4)}$ are all inequivalent since their characters are different as shown in the character table below. The two irreducible representations $X^{(5)}$ and $X^{(6)}$ of degree two are equivalent. All representations are displayed in Table 14 in Appendix B. The extended character t

3.2.2. The extra-special group $G_{0;2} = Q_8 = N_2$

The group $G_{0;2}$ is generated by 1, e_1 and e_2 with $e_1^2 = e_2^2 =$ 1; e_1

where $V^{(i)} = \text{spanf}\, u_i g;\ i = 1; :::; 8;$ are one-dimensional while

$$V^{(9)} = \text{spanf}\, u_9; u_{10} g; \quad V^{(10)} = \text{spanf}\, u_{11}; u_{12} g;$$

(33) $$V^{(11)} = \text{spanf}\, u_{13}; u_{14} g; \quad V^{(12)} = \text{spanf}\, u_{15}; u_{16} g$$

are two-dimensional subspaces carrying two pairwise equivalent representations according to Proposition 4, Theorem 10 and Theorem 11.

The basis vectors u_i are displayed in (47) in Appendix B. They have been found by using the above two algorithms.

Once the decomposition of $C[G_{3;0}]$ has been determined, one can compute all irreducible representations $X^{(i)}$ of $G_{3;0}$. The representations are displayed in Table 16 in Appendix B. The extended character table for $G_{3;0}$ is as follows:

(34)

char/class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$^{(1)}$	1	1	1	1	1	1	1	1	1	1
$^{(2)}$	1	1	1	1	1	1	1	1	1	1
$^{(3)}$	1	1	1	1	1	1	1	1	1	1
$^{(4)}$	1	1	1	1	1	1	1	1	1	1
$^{(5)}$	1	1	1	1	1	1	1	1	1	1
$^{(6)}$	1	1	1	1	1	1	1	1	1	1
$^{(7)}$	1	1	1	1	1	1	1	1	1	1
$^{(8)}$	1	1	1	1	1	1	1	1	1	1
$^{(9)}$	2	2	2i	2i	0	0	0	0	0	0
$^{(10)}$	2	2	2i	2i	0	0	0	0	0	0
$^{(11)}$	2	2	2i	2i	0	0	0	0	0	0
$^{(12)}$	2	2	2i	2i	0	0	0	0	0	0

Note that $X^{(9)} = X^{(12)}$ and $X^{(9)} = X^{(12)}$ since their characters are the same. To illustrate orthogonality of the characters, consider the inner product of the characters $^{(2)}$ and $^{(3)}$:

$$
\begin{aligned}
h^{(2)};\ ^{(3)} i &= \frac{1}{16} \sum_{i=1}^{10} jK_i j \ ^{(2)}_{K_i} \overline{^{(3)}_{K_i}} \\
&= \frac{1}{16}(1 \ (1)(1) + 1 \ (1)(1) + 1 \ (\ 1)(\ 1) + 1 \ (\ 1)(\ 1) \\
&\quad + 2 \ (1)(1) + 2 \ (1)(\ 1) + 2 \ (\ 1)(1) \\
&\quad + 2 \ (1)(\ 1) + 2 \ (\ 1)(1) + 2 \ (\ 1)(\ 1)) \\
&= 0:
\end{aligned}
\tag{35}
$$

which verifies the character orthogonality relation of the first kind. In a similar manner one can verify the character relation of the second kind.

Since the group $G_{1;2}$ belongs to the same class S_1 as $G_{3;0}$; it will not be discussed separately.

3.3.2. The group $G_{2;1} = \quad_1$

The group $G_{2;1}$ is generated by 1, e_1, e_2 and e_3 with $e_1^2 = e_2^2 = 1$ and $e_3^2 = \quad 1$; $e_i e_j = \quad e_j e_i$; $i \neq j$; while the group S S$_{16}$ isomorphic to $G_{2;1}$ is generated by the permutations of S_{16} as shown in Table 9 in Appendix A.

The decomposition of $C[G_{2;1}]$ looks the same as that of $C[G_{3;0}]$ displayed in (32), while the basis vectors u_i for this decomposition are displayed in (49) in Appendix B. They have been found by using the above two algorithms.

Once the decomposition of $C[G_{2;1}]$ has been found, one can compute all irreducible representations $X^{(i)}$ of $G_{2;1}$. The representations are displayed in Table 17 in Appendix B. The extended character table for $G_{2;1}$ is as follows:

char/class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
(1)	1	1	1	1	1	1	1	1	1	1
(2)	1	1	1	1	1	1	1	1	1	1
(3)	1	1	1	1	1	1	1	1	1	

The extended character table for $G_{0;3}$ is as follows:

(37)

char/class	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
(1)	1	1	1	1	1	1	1	1	1	1
(2)	1	1	1	1	1	1	1	1	1	1
(3)	1	1	1	1	1	1	1	1	1	1
(4)	1	1	1	1	1	1	1	1	1	1
(5)	1	1	1	1	1	1	1	1	1	1
(6)	1	1	1	1	1	1	1	1	1	1
(7)	1	1	1	1	1	1	1	1	1	1
(8)	1	1	1	1	1	1	1	1	1	1
(9)	2	2	2	2	0	0	0	0	0	0
(10)	2	2	2	2	0	0	0	0	0	0
(11)	2	2	2	2	0	0	0	0	0	0
(12)	2	2	2	2	0	0	0	0	0	0

Note that $X^{(9)} = X^{(11)}$ and $X^{(10)} = X^{(12)}$ since their characters are the same.

4. Conclusions

Due to the renewed interest in the relationship between nite Salingaros' vee groups $G = G_{p;q}$ and Cli ord algebras, the main goal of this paper has been to show how one can construct irreducible representations of these groups by decomposing their regular modules. In the process, two algorithms have been formulated which have allowed us to completely decompose regular modules of groups of order 4, 8, and 16 into irreducible

A. Images of the generators of the vee groups

In this Appendix, we show images of the generators of the vee groups $G_{p;q}$ for $p + q$ 3 in the symmetric groups S_n where $n = 2^{1+p+q}$:

Tab. 5: Generators for $G_{1;0}$ and $G_{0;1}$ in S_4

	$G_{1;0}$	Order	$G_{0;1}$	Order
1	$(1;2)(3;4)$			

Tab. 9: Generators for $G_{2,1}$ in S_{16}

	$G_{2,1}$	Order
1	(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)	2
e_1	(1, 3)(2, 4)(5, 9)(6, 10)(7, 11)(8, 12)(13, 15)(14, 16)	2
e_2	(1, 5)(2, 6)(3, 10)(4, 9)(7, 13)(8, 14)(11, 16)(12, 15)	2
e_3	(1, 7, 2, 8)(3, 12, 4, 11)(5, 14, 6, 13)(9, 15, 10, 16)	4

Tab. 10: Generators for $G_{1,2}$ in S_{16}

	$G_{1,2}$	Order
1	(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)	2
e_1	(1, 3)(2, 4)(5, 9)(6, 10)(7, 11)(8, 12)(13, 15)(14, 16)	2
e_2	(1, 5, 2, 6)(3, 10, 4, 9)(7, 13, 8, 14)(11, 16, 12, 15)	4
e_3	(1, 7, 2, 8)(3, 12, 4, 11)(5, 14, 6, 13)(9, 15, 10, 16)	4

Tab. 11: Generators for $G_{0,3}$ in S_{16}

	$G_{0,3}$	Order
1	(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)(13, 14)(15, 16)	2
e_1	(1, 3, 2, 4)(5, 9, 6, 10)(7, 11, 8, 12)(13, 15, 14, 16)	4
e_2	(1, 5, 2, 6)(3, 10, 4, 9)(7, 13, 8, 14)(11, 16, 12, 15)	4
e_3	(1, 7, 2, 8)(3, 12, 4, 11)(5, 14, 6, 13)(9, 15, 10, 16)	4

For the groups of order 4, all representations are inequivalent, and are shown in Tables 12 and 13.

In Table 12, the irreducible representations $X^{(i)}$ of $G_{1,0}$ are realized in irreducible $G_{1,0}$-invariant submodules of the group algebra $\mathbb{C}[G_{1,0}]$ which is decomposed as follows:

$$\mathbb{C}[G_{1,0}] = V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)}. \tag{38}$$

The one-dimensional submodules $V^{(i)}$ are spanned by the corresponding vectors u_i, $i = 1, \ldots, 4$. The coordinates of these vectors in the basis $B = \{1, -1, e_1, -e_1\}$ are as follows (:

$$\begin{aligned} V^{(1)} &= \mathrm{span}\{u_1\}, & u_1 &= (1, 1, 1, 1); \\ V^{(2)} &= \mathrm{span}\{u_2\}, & u_2 &= (1, -1, -1, 1); \\ V^{(3)} &= \mathrm{span}\{u_3\}, & u_3 &= (-1, 1, -1, 1); \\ V^{(4)} &= \mathrm{span}\{u_4\}, & u_4 &= (-1, -1, 1, 1). \end{aligned} \tag{39}$$

In Table 13, the irreducible representations $X^{(i)}$ of $G_{0,1}$ are realized in irreducible $G_{0,1}$-invariant submodules of the group algebra $\mathbb{C}[G_{0,1}]$ which is decomposed as

Tab. 12: Representations of $G_{1;0} = D_4$

	K_1	K_2	K_3	K_4
g	1	1	e_1	e_1
$X^{(1)}$	1	1	1	1
$X^{(2)}$	1	1	1	1
$X^{(3)}$	1	1	1	1
$X^{(4)}$	1	1	1	1

follows:

(40) $$C[G_{0;1}] = V^{(1)} \quad V^{(2)} \quad V^{(3)} \quad V^{(4)} :$$

are as follows:

$$V^{(1)} = \mathrm{span}\{u_1\}; \qquad u_1 = (1; 1; 1; 1; 1; 1; 1; 1);$$
$$V^{(2)} = \mathrm{span}\{u_2\}; \qquad u_2 = (\;\; 1; \;\; 1; 1; 1; 1; 1; \;\; 1; \;\; 1);$$
$$V^{(3)} = \mathrm{span}\{u_3\}; \qquad u_3 = (\;\; 1; \;\; 1; \;\; 1; \;\; 1; 1; 1; 1; 1);$$
$$V^{(4)} = \mathrm{span}\{u_4\}; \qquad u_4 = (1; 1; \;\; 1; \;\; 1; 1; 1; \;\; 1; \;\; 1);$$
$$V^{(5)} = \mathrm{span}\{u_5; u_6\}; \qquad u_5 = (\;\; 1; 1; \;\; 1; 1; \;\; 1; 1; 1; \;\; 1);$$
$$u_6 = (\;\; 5; 5; \;\; 5; 5; \;\; 1; 1; 1; \;\; 1);$$
$$V^{(6)} = \mathrm{span}\{u_7; u_8\}; \qquad u_7 = (1; \;\; 1; \;\; 1; 1; 0; 0; 0; 0);$$
$$u_8 = (1; \;\; 1; \;\; 1; 1; \;\; 1; 1; \;\; 1; 1): \qquad (43)$$

While the one-dimensional representations $X^{(1)}; X^{(2)}; X^{(3)}; X^{(4)}$ are inequivalent, the two-dimensional representations $X^{(5)}$ and $X^{(6)}$ are equivalent.

Tab. 14: Representations of $G_{2;0} = D_4 = N_1$

	K_1	K_2	K_3	K_4	K_5
g	1	1	e_1	e_2	e_{12}
$X^{(1)}$	1	1	1	1	1
$X^{(2)}$	1	1	1	1	1
$X^{(3)}$	1	1	1	1	1
$X^{(4)}$	1 !	1 !	1	1	1
$X^{(5)}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$	$\frac{3}{2}$		

are as follows:

$$
\begin{aligned}
&V^{(1)} = \mathrm{span}\{u_1\}; && u_1 = (1; 1; 1; 1; 1; 1; 1; 1);\\
&V^{(2)} = \mathrm{span}\{u_2\}; && u_2 = (\ \ 1;\ \ 1; 1; 1;\ \ 1;\ \ 1; 1; 1);\\
&V^{(3)} = \mathrm{span}\{u_3\}; && u_3 = (\ \ 1;\ \ 1;\ \ 1;\ \ 1; 1; 1; 1; 1);\\
&V^{(4)} = \mathrm{span}\{u_4\}; && u_4 = (1; 1;\ \ 1;\ \ 1;\ \ 1;\ \ 1; 1; 1);\\
&V^{(5)} = \mathrm{span}\{u_5; u_6\}; && u_5 = (0; 0; 0; 0;\ \ i; i;\ 1;\ \ 1);\\
& && u_6 = (\ \ i; i;\ \ 1; 1;\ \ i; i;\ 1;\ \ 1);\\
&V^{(6)} = \mathrm{span}\{u_7; u_8\}; && u_7 = (1;\ \ 1; i;\ \ i; 0; 0; 0; 0);\\
& && u_8 = (0; 0; 0; 0; 1;\ \ 1;\ \ i; i):
\end{aligned}
\tag{45}
$$

While the one-dimensional representations $X^{(1)}; X^{(2)}; X^{(3)}; X^{(4)}$ are inequivalent, the two-dimensional representations $X^{(5)}$ and $X^{(6)}$ are equivalent.

Tab. 15: Representations of $G_{0;2} = Q_8 = N_2$

	K_1	K_2	K_3	K_4	K_5
g	1	1	e_1	e_2	e_{12}
$X^{(1)}$	1	1	1	1	1
$X^{(2)}$	1	1	1	1	1
$X^{(3)}$	1	1	1	1	1
$X^{(4)}$	1	1	1	1	1
$X^{(5)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}i & 2i\\ 0 & i\end{pmatrix}$	$\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$	$\begin{pmatrix}i & 0\\ i & i\end{pmatrix}$
$X^{(6)}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$

In Table 16, the irreducible representations $X^{(i)}$ of $G_{3;0} = S_1$ are realized in irreducible $G_{3;0}$-invariant submodules of the group algebra $C[G_{3;0}]$ which is decomposed as follows:

$$
C[G_{3;0}] = \bigoplus_{i=1}^{12} V^{(i)}: \tag{46}
$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors u_i, $i = 1; \ldots; 16$; as shown below. The coordinates of these vectors in the standard basis

$$
B = \{1;\ \ 1; e_1;\ \ e_1; e_2;\ \ e_2; e_3;\ \ e_3; e_{12};\ \ e_{12}; e_{13};\ \ e_{13}; e_{23};\ \ e_{23}; e_{123};\ \ e_{123}\}
$$

Tab. 16: Part 2: Representations of $G_{3;0} = S_1$ for K_i; $i = 6;\ldots;10$

	K_6	K_7	K_8	K_9	K_{10}
g	e_2	e_3	e_{12}	e_{13}	e_{23}
$X^{(1)}$	1	1	1	1	1
$X^{(2)}$	1	1	1	1	1
$X^{(3)}$	1	1	1	1	1
$X^{(4)}$	1	1	1	1	1
$X^{(5)}$	1	1	1	1	1
$X^{(6)}$	1	1	1	1	1
$X^{(7)}$	1	1	1	1	1
$X^{(8)}$	1	1	1	1	1
$X^{(9)}$	0 i i 0	1 0 0 1	i 0 0 i	0 1 1 0	0 i i 0
$X^{(10)}$	0 1 1 0	0 i i 0	0 1 1 0	0 i i 0	i 0 0 i
$X^{(11)}$	0 1 1 0	0 i i 0	0 1 1 0	0 i i 0	i 0 0 i
$X^{(12)}$	0 i i 0	1 0 0 1	i 0 0 i	0 1 1 0	0 i i 0

While the representations $X^{(i)}$; $i = 1;\ldots;10$; are all inequivalent and irreducible, the remaining two-dimensional irreducible representations are equivalent as follows: $X^{(11)} = X^{(9)}$ and $X^{(12)} = X^{(10)}$.

In Table 18, the irreducible representations $X^{(i)}$ of $G_{0;3} = \;_2$ are realized in irreducible $G_{0;3}$-invariant submodules of the group algebra $C[G_{0;3}]$ which is decomposed as follows:

$$C[G_{0;3}] = \bigoplus_{i=1}^{12} V^{(i)}: \tag{50}$$

The submodules $V^{(i)}$ are spanned by the corresponding vectors u_i, $i = 1;\ldots;16$; as shown below. The coordinates of these vectors in the standard basis

$B = \{1$

Tab. 17: Part 1: Representations of $G_{2;1}$ = $_1$ for K_i; i = 1; : : : ; 5

	K_1	K_2	K_3	K_4	K_5
g	1	1	e_{123}	e_{123}	e_1
$X^{(1)}$	1	1	1	1	1
$X^{(2)}$	1	1	1	1	1
$X^{(3)}$	1	1	1	1	1
$X^{(4)}$	1	1	1	1	1
$X^{(5)}$	1	1	1	1	1
$X^{(6)}$	1	1	1	1	1
$X^{(7)}$	1	1	1	1	1
$X^{(8)}$	1 !	1 !	1 !	1 !	1 !
$X^{(9)}$	1 0 0 1 !	1 0 0 1 !	1 0 0 ! 1	1 0 0 1 !	1 0 0 1 !
$X^{(10)}$	1 0 0 1 !	1 0 0 1 !	1 0 0 1 !	1 0 0 ! 1	1 0 0 1 !
$X^{(11)}$	1 0 0 1 !	1 0 0 1 !	1 0 0 ! 1	1 0 0 1 !	1 0 0 1 !
$X^{(12)}$	1 0 0 1	1 0 0 1	1 0 0 1	1 0 0 1	1 0 0 1

are as follows:

$V^{(1)}$ = spanf u_1g; u_1 = (1 ; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(2)}$ = spanf u_2g; u_2 = (1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(3)}$ = spanf u_3g; u_3 = (1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(4)}$ = spanf u_4g; u_4 = (1 ; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(5)}$ = spanf u_5g; u_5 = (1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(6)}$ = spanf u_6g; u_6 = (1 ; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(7)}$ = spanf u_7g; u_7 = (1 ; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(8)}$ = spanf u_8g; u_8 = (1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1);

$V^{(9)}$ = spanf u_9; u_{10}g; u_9 = (1 ; 1; i; i; 0; 0; 0; 0; 0; 0; 0; 0; i; i; 1; 1);

u_{10} = (0 ; 0; 0; 0; 1; 1; i; i; i; i; 1; 1; 0; 0; 0; 0);

$V^{(10)}$ = spanf u_{11}; u_{12}g; u_{11} = (0 ; 0; 0; 0; i; i; 1; 1; 1; 1; i; i; 0; 0; 0; 0);

u_{12} = (1; 1; i; i; 0; 0; 0; 0; 0; 0; 0; 0; i; i; 1; 1);

$V^{(11)}$ = spanf u_{13}; u_{14}g; u_{13} = (0 ; 0; 0; 0; 1; 1; i; i; i; i; 1; 1; 0; 0; 0; 0);

u_{14} = (1 ; 1; i; i; 0; 0; 0; 0; 0; 0; 0; 0; i; i; 1; 1);

$V^{(12)}$ = spanf u_{15}; u_{16}g; u_{15} = (0 ; 0; 0; 0; 1; 1; i; i; i; i; 1; 1; 0; 0; 0; 0);

(51) u_{16} = (1; 1; i; i; 0; 0; 0; 0; 0; 0; 0; 0; i; i; 1; 1):

Tab. 18: Part 1: Representations of $G_{0;3} = \Omega_2$ for K_i; $i = 1; \ldots; 5$

	K_1	K_2	K_3	K_4	K_5

Tab. 18: Part 2: Representations of $G_{0;3} = \Omega_2$ for K_i; $i = 6; \ldots; 10$

	K_6	K_7	K_8	K_9	K_{10}
g	e_2	e_3	e_{12}	e_{13}	e_{23}
$X^{(1)}$	( )	( )	( )	( )	( )
$X^{(2)}$	( )	(−)	( )	(−)	(−)
$X^{(3)}$	(−)	( )	(−)	( )	(−)
$X^{(4)}$	(−)	(−)	(−)	(−)	( )
$X^{(5)}$	( )	( )	(−)	(−)	( )
$X^{(6)}$	( )				