DEPARTMENT OF MATHEMATICS
TECHNICAL REPORT

BILINEAR COVARIANTS AND SPINOR FIELD CLASSIFICATION IN QUANTUM CLIFFORD ALGEBRAS

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Abstract.

1. Introduction

The formalism of Cli ord algebras allows wide applications, in particular, the prominent construction of spinors and Dirac operators, and index theorems. Usually such algebras are essentially associated to an underlying quadratic vector space. Notwithstanding, there is nothing that complies to a symmetric bilinear form endowing the vector space [2]. For instance, symplectic Cli ord algebras are objects of huge interest. More generally, when one endows the underlying vector space with an arbitrary bilinear form, it evinces prominent features, especially regarding their representation theory. The most drastic character distinguishing the so called quantum and the orthogonal Cli ord algebras ones is that a di erent Z_n-grading arises, despite of the Z_2-grading being the same, since they are functorial. The most general Cli ord algebra

4. Classical spinors, algebraic spinors and spinor operato rs

Given an orthonormal basisf e g in

where $b^{0123} = p$. Now, the vector space isomorphisms

$$\mathcal{C}\ell^{+}_{1,3} \simeq \mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}\tfrac{1}{2}(1+e_0) \simeq \mathbb{C}^4 \simeq \mathbb{H}^2$$

give the equivalence among the classical, the operatorial, and the algebraic definitions of a spinor. In this sense, the spinor space $\mathbb{H}^2$ which carries the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representations of $SL(2,\mathbb{C})$ is isomorphic to the minimal left ideal $\mathcal{C}\ell_{1,3}\frac{1}{2}(1+e_0)$ – corresponding to the algebraic spinor – and also isomorphic to the even subalgebra $\mathcal{C}\ell^{+}_{1,3}$ – corresponding to the operatorial spinor. It is hence possible to write a Dirac spinor field as

$$\begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} [f] = \begin{pmatrix} q_1 & q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ q_2 & 0 \end{pmatrix}$$

$$\simeq \quad b + b^{23}$$

$$[(uvw)\ \rfloor]_B = uvw\ \ + uv(w\rfloor_A\ \ )\ \ \ uw(u\rfloor_g\ \ ) + w\wedge(u\rfloor_g(v\rfloor_A\ \ )) + u(v\rfloor_A(w\rfloor_B\ \ ))$$
$$v\wedge(u\rfloor_A w)\ \ + v\wedge w\wedge(u\rfloor_A\ \ )\ \ \ v\wedge(u\ _g\rfloor_A(w\rfloor_g\ \ )) + u\rfloor_A((v\rfloor_g w)\ \ )$$
$$v\wedge(u\rfloor_A(w\rfloor_B\ \ ))\ \ \ (w\rfloor_A u)v\ \ \ \ (v\rfloor_A w)u : \tag{19}$$

In (15) we used the minimal ideal provided by the idempotent

$$f\ = \tfrac{1}{4}(1 + \ _0)(1 + i\ \ _1\ _2) = \tfrac{1}{4}(1 + \ _0 + i\ \ _1\ _2 + i\ \ _0\ _1\ _2):$$

Now, in $C`(V;B)$ the formalism is recovered when we consider the idempotent

$$f_B = \tfrac{1}{4}(1 + \ _0 + i\ \ _{1_B}\ _2 + i\ \ _{0_B}\ _{1_B}\ _2) \tag{20}$$

where we let $\ _{1_B}\ _2 = (\ _1\ _2)_B$, $\ _{0_B}\ _{1_B}\ _2 = (\ _0\ _1\ _2)_B$; etc. in $C`(V;B)$. The formalism for $C`(V;B)$ is mutatis mutandis obtained, just by changing the standard Cli ord product to

$$\ _B\ = \ \ \ \ + A \tag{21}$$

The last expression is the prominent essence of transliterating $C`(V;B)$ to $C`(V;g)$. For instance, (15) evinces the necessity of de ning

$$f\ = \tfrac{1}{4}(1 + \ _0)(1 + i\ \ _1\ _2) \in C`(V;g): \tag{22}$$

Now, in $C\ \ C`^B_{1;3}$ we have

$$\begin{aligned} f_B &= \tfrac{1}{4}(1 + \ _0)_B(1 + i\ \ _{1_B}\ _2) \\ &= \tfrac{1}{4}(1 + \ _0)(1 + i\ \ _1\ _2) + \tfrac{i}{4}(A_{12} + A_{12}\ _0\ \ \ A_{02}\ _1 + A_{01}\ _2): \end{aligned} \tag{23}$$

Herein we shall denote

$$f_B = f\ + f(A) \tag{24}$$

where $f(A) = \tfrac{i}{4}(A_{12} + A_{12}\ _0\ \ \ A_{02}\ _1 + A_{01}\ _2)$.

In the Dirac representation (A.3), the idempotent f in (22) reads

$$f\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and as

$$\ _{0_B}\ _{1_B}\ _2 = \ \ _0\ _1\ _2 + A_{01}\ _2\ \ \ A_{02}\ _1 + A_{12}\ _0; \tag{25}$$

one can substitute it in (24) to obtain

$$f_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2iA_{12} & 0 & 0 & iA_{02}\ \ A_{01} \\ 0 & 2iA_{12} & iA_{02} + A_{01} & 0 \\ 0 & iA_{02} + A_{01} & 0 & 0 \\ iA_{02}\ \ A_{01} & 0 & 0 & 0 \end{pmatrix} : \tag{26}$$

When $A = 0$ it implies that $B = g$ and the standard spinor formalism is recovered. Let us denote by $C\ell^{B}_{1,3}$ the Clifford algebra $C\ell(V, B)$, where $V = \mathbb{R}^4$ and $B = \ + A$, where denotes the Minkowski metric.

An arbitrary element of $C\ell^{B}_{1,3}$ is written as

$$_B = c + c \quad + c \ (\quad)_B + c \ (\quad)_B + p(\ _0\ _1\ _2\ _3)_B : \qquad (27)$$

By using (21, 25), (27) reads

$$_B = \quad + c \ A \quad + c \ (A \quad + A \quad + A \quad) + p \quad A \ (\quad + A \ ) \qquad (28)$$

where is an element in the standard Clifford algebra $C\ell_{1,3}$ of the form given by (6).

Herein we shall rewrite (28) as

$$_B = \quad + \quad (A)$$

$2_B)$ $\sigma_B \neq 0$; $\omega_B = 0$.

$3_B)$ $\sigma_B = 0$; $\omega_B \neq 0$.

$4_B)$ $\sigma_B = 0 = \omega_B$; $K_B \neq 0$; $S_B \neq 0$.

$5_B)$ $\sigma_B = 0 = \omega_B$; $K_B = 0$; $S_B \neq 0$.

$6_B)$ $\sigma_B = 0 = \omega_B$; $K_B \neq 0$; $S_B = 0$.

It is always possible to write:

$$\sigma_B = \sigma + \sigma(A); \tag{32}$$

$$J_B = J + J(A); \tag{33}$$

$$S_B = S + S(A); \tag{34}$$

$$K_B = K + K(A); \tag{35}$$

$$\omega_B = \omega + \omega(A). \tag{36}$$

In general, since we assume $A \neq 0$ (otherwise there is nothing new to prove, as when $A = 0$ it implies that $C\ell(V;B) = C\ell(V;g)$), it follows that all the A-dependent quantities $\sigma(A)$, $J(A)$, $S(A)$,

Table 1. Correspondence among the spinor field and the (quantum) B-spinor fields under Lounesto spinor field classification.

	Quantum Spinor Fields	Spinor Fields	
type-(1_B)	B-Dirac	Dirac	type-(1)
		Dirac	type-(2)
		Dirac	type-(3)
		Flag-dipoles	type-(4)
		Flagpoles (also Elko, Majorana)	type-(5)
		Weyl	type-(6)
type-(2_B)	B-Dirac	Dirac	type-(3)
		Dirac	type-(1)
type-(3_B)	B-Dirac	Dirac	type-(2)
		Dirac	type-(1)
type-(4_B)	B-flag-dipole	Dirac	type-(1)
type-(5_B)	B-flagpole	Dirac	type-(1)
type-(6_B)	B-Weyl	Dirac	type-(1)

condition $\omega_B = \omega + \omega(A) \neq 0$ must hold. It is tantamount to assert that $0 \neq \omega \neq -\omega(A)$.

ii) $\sigma \neq 0$ and $\omega \neq 0$. This case corresponds to the type-(1) Dirac spinor fields Here both the conditions $0 \neq \omega \neq -\omega(A)$ and $0 \neq \sigma \neq -\sigma(A)$ has to hold.

3_B) $\sigma_B = 0$; $\omega_B \neq 0$. Despite the condition $\omega_B \neq 0$ is compatible to both the possibilities $\omega = 0$ and $\omega \neq 0$ (clearly the condition $\omega \neq 0$ is compatible to $\omega_B \neq 0$ if $\omega \neq -\omega(A)$), the condition $\sigma_B = 0$ implies that $\sigma = -\sigma(A)$, which does not equal zero. To summarize:

i) $\omega = 0$ and $\sigma \neq 0$. This case corresponds to the type-(2) Dirac spinor fields The condition $\omega = 0$ is compatible to $\omega_B \neq 0$, but as $\sigma \neq 0$, the additional condition $\sigma_B = \sigma + \sigma(A) \neq 0$ must be imposed. Equivalently, $0 \neq \sigma \neq -\sigma(A)$.

ii) $\sigma \neq 0$ and $\omega \neq 0$. This case corresponds to the type-(1) Dirac spinor fields Here both the conditions $0 \neq \omega \neq -\omega(A)$ and $0 \neq \sigma \neq -\sigma(A)$ must be imposed.

4_B) $\sigma_B = 0 = \omega_B$; $K_B \neq 0$; $S_B \neq 0$.

5_B) $\sigma_B = 0 = \omega_B$; $K_B = 0$; $S_B \neq 0$.

6_B) $\sigma_B = 0 = \omega_B$; $K_B \neq 0$; $S_B = 0$.

All the quantum spinor fields 4_B), 5_B), and 6_B) are defined by the condition $\sigma_B = 0 = \omega_B$. This implies that $\sigma = -\sigma(A) \neq 0$, and that $\omega = -\omega(A) \neq 0$. It means that all the singular B-spinor fields correspond to (p)1.94942(i)-9.00463(n)16468(n)1.[illegible]327.696(s)49532 TJ 11

-0.64498(e)s

type-(1_B) Dirac spinor fields correspond to all spinor fields in the orthogonal Clifford algebra. A deep discussion about these results is going to be accomplished in the next Section.

7. Concluding remarks and outlook

The mathematical apparatus provided by the quantum Clifford algebraic formalism is a powerful tool, in particular to bring additional interpretations about the underlying standard spacetime structures. For instance, equations (23–36) illustrate that the distribution of intrinsic angular momentum, formerly a legitimate bivector in the standard Clifford algebra $C\ell(V;g)$, is now the direct sum of a bivector and a scalar when considered in $C\ell(V;B)$ from the point of view of $C\ell(V;g)$, evincing the different $\mathbb{Z}_n$-grading induced by the antisymmetric part of the arbitrary bilinear form B. Furthermore, now, the bilinear covariant K is a paravector – the sum of a vector and a scalar – which is not a homogeneous Clifford element. Indeed, in $C\ell(V;B)$ it is a homogeneous 1-form, but in $C\ell(V;g)$ it is a paravector.

Some questions and possible answers can still be posed in the context of the quantum Clifford algebraic arena. The mathematical formali

(5) spinor fields is a prime candidate to describe the dark matter [19, 20]. In particular,

$$e_{1_B} e_{3_B} f_B = \tfrac{1}{4}((iA_{23} \quad A_{13})1 + (\,iA_{23} \quad A_{13})e_0 + A_{03}e_1 \quad iA_{03}e_2 \quad (A_{01} \quad iA_{02})e_3$$
$$e_{13} + ie_{23} \quad e_{013} + ie_{023});$$
$$e_{3_B} f_B = \tfrac{1}{4}(\quad (A_{03} + iA_{03}A_{12} + iA_{01}A_{23} \quad iA_{13}A_{02})1 + iA_{23}e_1 \quad iA_{13}e_2 +$$
$$(1 + iA_{12})e_3 \quad iA_{23}e_{01} + iA_{13}e_{02} \quad (1 + iA_{12})e_{03} \quad iA_{03}e_{12} +$$
$$iA_{02}e_{13} \quad iA_{01}e_{23} \quad ie_{0123} + ie_{123});$$
$$e_{1_B} f_B = \tfrac{1}{4}(\quad (A_{01} \quad iA_{02})1 + e_1 \quad ie_2 \quad e_{01} + ie_{02}); \qquad \text{(A.9)}$$

where 1 denotes the unity of $C \quad C`^B_{1;3}$: Of course, when we set $A_{ij} = 0$ for all the coe cients of the antisymmetric part A appearing in (A.9), we obtain back the explicit basis for the ideal $S = (C \quad C`_{1;3})f$ shown in (A.1). Due to the relations (A.6), the gamma matrices (A.3) also represent the generators $e_0; e_1; e_2; e_3$ in the faithful and irreducible representation of the algebra $C \quad C`^B_{1;3}$ in the ideal S_B. This can be checked directly by computing these matrices in the explicit symbolic basis (A.9) with CLIFFORD [22].

Appendix B. Additional terms in the quantum spinor elds

Recall from (31) that a B-spinor has the form

$$(\ _B)_B(f_B) = (\ )_B f + \quad (A)_B f + (\ )_B f(A) + (\quad (A))_B f(A): \qquad \text{(B.1)}$$

where the term $(\ )_B f$ is the classical spinor eld displayed in (15). The remaining terms in the above expression represent correction terms and are provided by:

(a) The term $\boxed{4i(\quad (A))_B f(A)}$ is given by

$$p \quad b^{013}(A_{01}(A_{01}A_{32} + A_{20}A_{31} + A_{12}A_{30}) + A_{12}A_{13} + A_{03}A_{20})$$
$$+ b^{023}(A_{02}(A_{01}A_{32} + A_{20}A_{31} + A_{12}A_{30} + A_{30}) + A_{12}A_{13} + A_{03}A_{20})$$
$$+ b^{123}(A_{12}(A_{01}A_{32} + A_{20}A_{31} + A_{12}A_{30}) \quad A_{23}A_{20} \quad A_{31}A_{01})$$
$$+ b^{012}(A_{10}A_{01} + A_{20}A_{02} + A_{12}A_{12})$$
$$+ \ _0[p(A_{13}A_{01} \quad A_{23}A_{20} + 2A_{12}A_{12}A_{13} + A_{23}A_{20}A_{12} + A_{23}A_{01}A_{12}) + sA_{12}]$$
$$+ \ _1[p(A_{12}A_{13} \quad A_{12}A_{23} \quad A_{03}A_{01} + A_{01}A_{20}A_{32} + A_{01}A_{20}A_{13} + A_{02}A_{12}A_{03}$$
$$+ A_{03}A_{12}A_{21} + A_{23}A_{01}A_{10}) + sA_{01}]$$
$$+ \ _2[p(A_{03}A_{20} + A_{01}A_{01}A_{32} + A_{13}A_{01}A_{02} + A_{13}A_{20}A_{02}) + sA_{02}]$$
$$+ \ _3[p(A_{01}A_{01} + A_{02}A_{02} + A_{02}A_{12}A_{13} + A_{12}A_{20}A_{10} + A_{02}A_{20}A_{12} + A_{01}A_{12}A_{13})]$$
$$+ \ _{01}\ p \quad b^{013}(A_{13}A_{20} + A_{21}A_{30}) + b^{023}(A_{03}A_{12} + A_{13}A_{20}) \quad b^{123}A_{23}A_{12}$$
$$+ \ _{02}\ p \quad b^{013}A_{13}A_{01} + b^{023}A_{13}A_{01} + b^{123}A_{13}A_{21}$$
$$+ \ _{03}\ p \quad b^{013}A_{01}A_{12} + b^{023}A_{02}A_{12} + b^{123}A_{12}A_{21}$$

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$$+ \ _{12} \quad A_{01}A_{12} + \quad A\ _{13}A_{01} + \quad (A_{13}A_{20} + A_{13}A$$

$+ \;_{23}\; p \; b^{013}A_{01}A_{10} + b^{023}A_{01}A_{20} + b^{123}A_{12}A_{10}$

$+ \;_{023}A_{12}A_{01}$

$+ \;_{031}($

$$
\begin{aligned}
&+ b^{03}(A_{02}A_{31} + A_{23}A_{01} + A_{30}) + b^{31}(A_{21}A_{31} + A_{23}) \\
&+ p(A_{30}A_{20} + A_{32}A_{01} + A_{21}A_{20}A_{13} + A_{30}A_{21}A_{12} + A_{30}A_{20} \\
&\qquad + A_{32}A_{01} + A_{21}A_{12}(A_{02} + A_{30}) + b^{023}A_{23}A_{21} \\
&+ b^{12}(A_{21}A_{12} \quad 1) + b^{23}(A_{31} \quad A_{32}A_{21}) \\
+ \ {}_{1} \ & b^{01}(A_{20}A_{10} + A_{12} + 2) + b^{02}(A_{20}A_{02} \quad 1) + b^{03}(A_{32} \quad A_{30}A_{02}) \\
&\qquad + b^{31}(A_{01}A_{32} + A_{12}A_{30} + A_{30}) \\
&+ b^{12}(A_{20}A_{12} + A_{01} + A_{02}) + b^{23}(A_{30} + A_{32}A_{20}) \\
&+ p(A_{32}A_{01}A_{02} + A_{30}A_{20}A_{12})] \\
+ \ {}_{2} \ & b^{01}(A_{10}A_{01} \quad 1) + b^{02}(A_{20}A_{01} + A_{12} + 2) + b^{03}(A_{30}A_{01} + A_{13}) \ A A_{20} + 2) + b^{02}
\end{aligned}
$$

$$
\begin{aligned}
&+ {}_{3}\ A_{20}A_{02} + A_{10}A_{01} + A_{12}A_{12} + b^{03}A_{21} + b^{13}(A_{20} + A_{01}A_{21}) + b^{23}A_{01} \\
&+ {}_{01}\ b^{013}\ \ A_{03}A_{12} + A_{31}A_{20}) + b^{123}A_{23}(A_{12} + A_{02}\ \ + b^{012}(A_{10} + A_{20}A_{21}) \\
&\qquad + b^{013}A_{32}A_{01} + b^{0}A_{20} + b^{1}A_{21} \\
&+ {}_{02}\ b^{013}A_{01}A_{31} + b^{123}A_{32}A_{01} + b^{012}(A_{20} + A_{21}A_{01}) \\
&\qquad + b^{023}(A_{23}A_{01} + A_{20}A_{31} + A_{30}A_{12}) + b^{0}A_{01} + b^{2}A_{01} \\
&+ {}_{03}\ b^{012}A_{20} \\
&+ {}_{12}\ b^{013}A_{01}A_{31} + b^{123}(A_{20}A_{31} + A_{30}A_{12} + A_{32}A_{01}) \\
&\qquad + b^{023}(A_{13}A_{01} + A_{03}A_{20}) + b^{1}A_{01} + b^{2}A_{02} \\
&+ {}_{13}\ b^{013}A_{12} + b^{123}A_{01} + b^{013}A_{02} + b^{3}A_{02} \\
&+ {}_{23}\ b^{123}(A_{01}(A_{02} + A_{21}) + A_{20}) + b^{023}A_{12} \qquad \text{(B.5)}
\end{aligned}
$$

References

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