DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

SQUARE ROOTS OF -1 IN REAL CLIFFORD ALGEBRAS

E. HITZER, J. HELMSTETTER and R. ABLAMOWICZ

April 2012

No. 2012-3

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

Square Roots of −1 in Real Clifford Algebras

Eckhard Hitzer, Jacques Helmstetter and Rafal Ablamowicz

Abstract. It is well known that Clifford (geometric) algebra offers a geometric interpretation for square roots of −1 in the form of blades that square to minus 1. This extends to a geometric interpretation of quaternions as the side face bivectors of a unit cube. Systematic research has been done [32] on the biquaternion roots of −1, abandoning the restriction to blades. Biquaternions are isomorphic to the Clifford (geometric) algebra $C\ell(3,0)$ of $\mathbb{R}^3$. Further research on general algebras $C\ell(p,q)$ has explicitly derived the geometric roots of −1 for $p+q \leq 4$ [17]. The current research abandons this dimension limit and uses the Clifford algebra to matrix algebra isomorphisms in order to algebraically characterize the continuous manifolds of square roots of −1 found in the different types of Clifford algebras, depending on the type of associated ring ($\mathbb{R}$, $\mathbb{H}$, $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$). At the end of the paper explicit computer generated tables of representative square roots of −1 are given for all Clifford algebras with $n = 5, 7$, and $s = 3 \pmod 4$ with the associated ring $\mathbb{C}$. This includes, e.g., $C\ell(0,5)$ important in Clifford analysis, and $C\ell(4,1)$ which in applications is at the foundation of conformal geometric algebra. All these roots of −1 are immediately useful in the construction of new types of geometric Clifford Fourier transformations.

Mathematics Subject Classification (2010). Primary 15A66; Secondary 11E88, 42A38, 30G35.

Keywords. algebication will be available at
/4961 as part of E. Hitzer, S. Sangwine (eds.), "Quater-
orms and wavelets", Trends in Mathematics, Birkhauser,

1. Introduction

The young London Goldsmid professor of applied mathematics W. K. Clifford created his *geometric algebras*[1] in 1878 inspired by the works of Hamilton on quaternions and by Grassmann's exterior algebra. Grassmann invented the antisymmetric outer product of vectors, that regards the oriented parallelogram area spanned by two vectors as a new type of number, commonly called bivector. The bivector represents its own plane, because outer products with vectors in the plane vanish. In three dimensions the outer product of three linearly independent vectors defines a so-called trivector with the magnitude of the volume of the parallelepiped spanned by the vectors. Its orientation (sign) depends on the handedness of the three vectors.

In the Clifford algebra [13] of $\mathbb{R}^3$ the three bivector side faces of a unit cube $\{e_1e_2, e_2e_3, e_3e_1\}$ oriented along the three coordinate directions $\{e_1, e_2, e_3\}$ correspond to the three quaternion units i, j, and k. Like quaternions, these three bivectors square to minus one and generate the rotations in their respective planes.

Beyond that Clifford algebra allows to extend complex numbers to higher dimensions [4, 14] and systematically generalize our knowledge of complex numbers, holomorphic functions and quaternions into the realm of Clifford analysis. It has found rich applications in symbolic computation, physics, robotics, computer graphics, etc. [5, 6, 9, 11, 23]. Since bivectors and trivectors in the Clifford algebras of Euclidean vector spaces square to minus one, we can use them to create new geometric kernels for Fourier transformations. This leads to a large variety of new Fourier transformations, which all deserve to be studied in their own right [6, 10, 15, 16, 19, 20, 22, 25–29, 31].

In our current research we will treat square roots of -1 in Clifford algebras $C\ell(p,q)$ of both Euclidean (positive definite metric) and non-Euclidean (indefinite metric) non-degenerate vector spaces, $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $\mathbb{R}^{p,q}$, respectively. We know from Einstein's special theory of relativity that non-Euclidean vector spaces are of fundamental importance in nature [12]. They are further, e.g., used in computer vision and robotics [9] and for general algebraic solutions to contact problems [23]. Therefore this chapter is about characterizing square roots of -1 in all Clifford algebras $C\ell(p,q)$, extending previous limited research on $C\ell(3,0)$ in [32] and $C\ell(p,q)$, $n = p+q \leq 4$ in [17]. The manifolds of square roots of -1 in $C\ell(p,q)$, $n = p+q = 2$, compare Table 1 of [17], are visualized in Fig. 1.

First, we introduce necessary background knowledge of Clifford algebras and matrix ring isomorphisms and explain in more detail how we will characterize and classify the square roots of -1 in Clifford algebras in Section 2. Next, we treat section by section (in Sections 3 to 7) the square roots of -1 in Clifford algebras which are isomorphic to matrix algebras with associated

---

[1] In his original publication [8] Clifford first used the term "geometric algebras". Subsequently in mathematics the new term "Clifford algebras" [24] has become the proper mathematical term. For emphasizing the "geometric" nature of the algebra, some researchers continue [6, 13, 14] to use the original term geometric algebra(s).

$\mathrm{Cent}(f) \bigcap G(A)$ is contained in the neutral[7] connected component of $G(A)$, and the dimension of its conjugacy class is

$$\dim(A) - \dim(\mathrm{Cent}(f)). \tag{2.1}$$

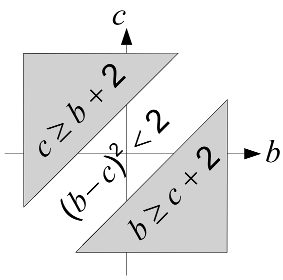
Note that for invertible $g \in \mathrm{Cent}(f)$ we have $g^{-1} f g = f$.

Besides, let $Z(A)$ be the center of A, and let $[A; A]$ be the subspace instance, ω (mentioned above) and ω are central square roots of 1 in $M(2d; C)$ which constitute two conjugacy classes of dimension 0. Obviously, $\mathrm{Spec}(\omega) = 1$.

are related as follows: $\mathrm{tr}(f) = 2d\,\mathrm{Scal}(f)$. If f is a square root of -1, it turns V into a vector space over $\mathbb{C}$ (if the complex number i operates like f on V). If $(e_1, e_2, \ldots, e_d)$ is a $\mathbb{C}$-basis of V, then $(e_1, f(e_1), e_2, f(e_2), \ldots, e_d, f(e_d))$ is a $\mathbb{R}$-basis of V, and the $2d \times 2d$ matrix of f in this basis is

$$\mathrm{diag}\underbrace{\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)}_{d} \tag{3.2}$$

Consequently all square roots of -1 in A are conjugate. The centralizer of a square root f of -1 is the algebra of all $\mathbb{C}$-linear endomorphisms g of V (since i operates like f on V). Therefore, the $\mathbb{C}$-dimension of $\mathrm{Cent}(f)$ is d^2 and its $\mathbb{R}$-dimension is $2d^2$. Finally, the dimension (2.1) of the conjugacy class of f is $\dim(A) - \dim(\mathrm{Cent}(f)) = 4d^2 - 2d^2 = 2d^2 = \dim(A)/2$. The two connected components of $G(A)$ are determined by the sign of the determinant. Because of the next lemma, the $\mathbb{R}$-determinant of every element of $\mathrm{Cent}(f)$ is ≥ 0. Therefore, the intersection $\mathrm{Cent}(f) \cap G(A)$ is contained in the neutral connected component of $G(A)$ and, consequently, the conjugacy class of f has two connected components like $G(A)$. Because of the next lemma, the $\mathbb{R}$-trace quare



c
b
c ≥ b + 2
(b − c)² < 2
b ≥ c + 2

A (with $f, f' \in M(2d, \mathbb{R})$) has a determinant in $\mathbb{R}^2$ which is obviously $(\det(f), \det(f'))$, and the four connected components of $G(A)$ are determined by the signs of the two components of $\det_{\mathbb{R}^2}(f, f')$.

The lowest dimensional example ($d = 1$) is $C\ell(2,1)$ isomorphic to $M(2, \mathbb{R}^2)$. Here the pseudoscalar $\omega = e_{123}$ has square $\omega^2 = +1$. The center of the algebra is $\{1, \omega\}$ and includes the idempotents $\epsilon_\pm = (1 \pm \omega)/2$, $\epsilon_\pm^2 = \epsilon_\pm$, $\epsilon_+\epsilon_- = \epsilon_-\epsilon_+ = 0$. The basis of the algebra can thus be written as $\{\epsilon_+, e_1\epsilon_+, e_2\epsilon_+, e_{12}\epsilon_+, \epsilon_-, e_1\epsilon_-, e_2\epsilon_-, e_{12}\epsilon_-\}$, where the first (and the last) four elements form a basis of the subalgebra $C\ell(2,0)$ isomorphic to $M(2, \mathbb{R})$. In terms of matrices we have the identity matrix $(1, 1)$ representing the scalar part, the idempotent matrices $(1, 0)$, $(0, 1)$, and the ω matrix $(1, -1)$, with 1 the unit matrix of $M(2, \mathbb{R})$.

The square roots of $(-1, -1)$ in A are pairs of two square roots of -1 in $M(2d, \mathbb{R})$. Consequently they constitute a unique conjugacy class with four connected components of dimension $4d^2 = \dim(A)/2$. This number can be obtained in two ways. First, since every element $(f, f') \in A$ (with $f, f' \in M(2d, \mathbb{R})$) has twice the dimension of the components $f \in M(2d, \mathbb{R})$ of Section 3, we get the component dimension $2 \cdot 2d^2 = 4d^2$. Second, the centralizer $\mathrm{Cent}(f, f')$ has twice the dimension of $\mathrm{Cent}(f)$ of $M(2d, \mathbb{R})$, therefore $\dim(A) - \mathrm{Cent}(f, f') = 8d^2 - 4d^2 = 4d^2$. In the above example for $d = 1$ the four components are characterized according to (3.5) by the values of the coefficients of $\omega_1 e_{12}\epsilon_+$ and $\omega_1' e_{12}\epsilon_-$ as

$c_1: \quad \omega_1 \geq 1, \quad \omega_1' \geq 1,$

$c_2: \quad \omega_1 \geq 1, \quad \omega_1' \leq -1,$

$c_3: \quad \omega_1 \leq -1, \quad \omega_1' \geq 1,$

whence $\mathrm{Scal}(f, f') = \mathrm{Spec}(f, f') = 0$ if (f, f') is a square root of $(-1, -1)$, compare with (5.2).

The group $\mathrm{Aut}(A)$ has two[17] connected components; the neutral component is $\mathrm{Inn}(A)$, and the other component contains the swap automorphism $(f, f') \mapsto (f', f)$.

The simplest example is $d = 1$, $A = \mathbb{H}^2$, where we have the identity pair $(1, 1)$ representing the scalar part, the idempotents $(1, 0)$, $(0, 1)$, and ω as the pair $(1, -1)$.

$A = \mathbb{H}^2$ is isomorphic to $C\ell(0, 3)$. The pseudoscalar $\omega = e_{123}$ has the square $\omega^2 = +1$. The center of the algebra is $\{1, \omega\}$, and includes the idempotents $\epsilon_\pm = \frac{1}{2}(1 \pm \omega)$, $\epsilon_\pm^2 = \epsilon_\pm$, $\epsilon_+\epsilon_- = \epsilon_-\epsilon_+ = 0$. The basis of the algebra can thus be written as $\{\epsilon_+, e_1\epsilon_+, e_2\epsilon_+, e_{12}\epsilon_+, \epsilon_-, e_1\epsilon_-, e_2\epsilon_-, e_{12}\epsilon_-\}$ where the first (and the last) four elements form a basis of the subalgebra $C\ell(0, 2)$ isomorphic to $\mathbb{H}$.

7. Square roots of -1 in $\mathcal{M}(2d, \mathbb{C})$

The lowest dimensional example for $d = 1$ is the Pauli matrix algebra $A = \mathcal{M}(2, \mathbb{C})$ isomorphic to the geometric algebra $C\ell(3, 0)$ of the 3D Euclidean space and $C\ell(1, 2)$. The $C\ell(3, 0)$ vectors e_1, e_2, e_3 correspond one-to-one to the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.1}$$

with $\sigma_1\sigma_2 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The element $\omega = \sigma_1\sigma_2\sigma_3 = i1$ represents the central pseudoscalar e_{123} of $C\ell(3, 0)$ with square $\omega^2 = -1$. The Pauli algebra has the following idempotents

$$\epsilon_1 = \epsilon_1^2 = 1, \qquad \epsilon_0 = \frac{1}{2}(1 + \sigma_3), \qquad \epsilon_{-1} = 0. \tag{7.2}$$

The idempotents correspond via

$$f = i(2\epsilon - 1), \tag{7.3}$$

to the square roots of -1:

$$f_1 = i1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}; \; f_0 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \; f_{-1} = -i1 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}; \tag{7.4}$$

where by complex conjugation $f_{-1} = \overline{f_1}$. Let the idempotent $\epsilon_0' = \frac{1}{2}(1 - \sigma_3)$ correspond to the matrix $f_0' = -i\sigma_3$. We observe that f_0 is conjugate to $f_0' = \sigma_1^{-1} f_0 \sigma_1 = -\sigma_1\sigma_2 = -f_0$ using $\sigma_1^{-1} = \sigma_1$ but f_1 is not conjugate to f_{-1}. Therefore, only f_1, f_0, f_{-1} lead to three distinct conjugacy classes of square roots of -1 in $\mathcal{M}(2, \mathbb{C})$. Compare Appendix B for the corresponding computations with CLIFFORD for Maple.

[17] Compare Footnote 14.

In general, if $A = M(2d, \mathbb{C})$, then $\dim(A) = 8d^2$. The group $G(A)$ has one connected component. The square roots of -1 in A are in bijection with the idempotents ε [2] according to (7.3). According[18] to (7.3) and its inverse $\varepsilon = \frac{1}{2}(1 - if)$ the square root of -1 with $\mathrm{Spec}(f_-) = k/d = -1$, i.e. $k = -d$ (see below), always corresponds to the trival idempotent $\varepsilon_- = 0$, and the square root of -1 with $\mathrm{Spec}(f_+) = k/d = +1$, $k = +d$, corresponds to the identity idempotent $\varepsilon_+ = 1$.

If f is a square root of -1, then $V = \mathbb{C}^{2d}$ is the direct sum of the eigenspaces[19] associated with the eigenvalues i and $-i$. There is an integer k such that the dimensions of the eigenspaces are respectively $d + k$ and $d - k$. Moreover, $-d \le k \le d$. Two square roots of -1 are conjugate if and only if they give the same integer k. Then, all elements of $\mathrm{Cent}(f)$ consist of diagonal block matrices with 2 square blocks of $(d + k) \times (d + k)$ matrices and $(d - k) \times (d - k)$ matrices. Therefore, the $\mathbb{C}$-dimension of $\mathrm{Cent}(f)$ is $(d + k)^2 + (d - k)^2$. Hence the $\mathbb{R}$-dimension (2.1) of the conjugacy class of f:

$$8d^2 - 2(d + k)^2 - 2(d - k)^2 = 4(d^2 - k^2). \tag{7.5}$$

Also, from the equality $\mathrm{tr}(f) = (d + k)i$

with an integer k other than 0 is mapped by complex conjugation to the class associated with k. In particular the complex conjugation maps the class f ! g (associated with $k = d$) to the class f ! g associated with $k = d$.

All these observations can easily veri ed for the above example of $d = 1$ of the Pauli matrix algebra $A = M(2; C)$. For $d = 2$ we have the isomorphism of $A = M(4; C)$ with $C`(0; 5)$, $C`(2; 3)$ and $C`(4; 1)$. While $C`(0; 5)$ is important in Cli ord analysis, $C`(4; 1)$ is both the geometric algebra of the Lorentz space $R^{4;1}$ and the conformal geometric algebra of 3D Euclidean

! = e_{12345}: Again, Spec(

where $n_1 + n_2 = 2d = 8$ and $n_1 = d + k = 4 + k$ and $n_2 = d \quad k = 4 \quad k$. The ordinary root of 1 corresponds to $k = 0$ whereas the exceptional roots correspond to $k \neq 0$:

1. When $k = 4$; we have $_4(t) = (t \quad i)^8$; $m_4(t) = t \quad i$; and $F_4 = \mathrm{diag}(\overbrace{i, \ldots, i}^{8})$ which in the representation used by CLIFFORD [3] corresponds to the non-trivial central element $f_4 = ! = e_{1234567}$: Clearly, $\mathrm{Spec}(f_4) = 1 = \frac{k}{d}$; $\mathrm{Scal}(f_4) = 0$; the C-dimension of the centralizer $\mathrm{Cent}(f_4)$ is 64; and the R-dimension of the conjugacy class of f_4 is zero since f_4 2 $Z(A)$: Thus, the R-dimension of the class is again zero in agreement with (7.5).
2. When $k = \quad 4$; we have $_4(t) = (t + i)^8$; $m \; _4(t) = t + i$; and $F \; _4 = \mathrm{diag}(\overbrace{\quad i, \ldots, \quad i}^{8})$ which corresponds to $f \; _4 = \quad ! = \quad e_{1234567}$: Again, $\mathrm{Spec}(f \; _4) = \quad 1 = \frac{k}{d}$; $\mathrm{Scal}(f \; _4) = 0$; the C-dimension of the centralizer Cent(f) is 64 and the conjugacy class of f $_4$ con œ

When $k = -1$, then $\alpha_{-1}(t) = (t-i)^3(t+i)^5$ and $\mu_{-1}(t) = (t-i)(t+i)$.

Then the root $F_{-1} = \mathrm{diag}(i, i, i, \overbrace{-i, \ldots, -i}^{5})$ corresponds to

$$f_{-1} = \frac{1}{4}(e_{23} - e_{45} + 3e_{67} + e_{123} - e_{145} - e_{167} - e_{234567} - e_{1234567}). \quad (7.18)$$

Thus, $\mathrm{Spec}(f_{-1}) = -\frac{1}{4} = \frac{k}{d}$ and so f_{-1} is another exceptional root.

When $k = -2$, then $\alpha_{-2}(t) = (t-i)^2(t+i)^6$ and $\mu_{-2}(t) = (t-i)(t+i)$.

Then the root $F_{-2} = \mathrm{diag}(i, i, \overbrace{-i, \ldots, -i}^{6})$ corresponds to

$$f_{-2} = \frac{1}{2}(e_{67} - e_{45} + e_{123} - e_{1234567}). \quad (7.19)$$

Since $\mathrm{Spec}(f_{-2}) = -\frac{1}{2} = \frac{k}{d}$, we see that f_{-2} is also an exceptional root.

When $k = -3$, then $\alpha_{-3}(t) = (t-i)(t+i)^7$ and $\mu_{-3}(t) = (t-i)(t+i)$.

Then the root $F_{-3} = \mathrm{diag}(i, \overbrace{-i, \ldots, -i}^{7})$ corresponds to

$$f_{-3} = \frac{1}{4}(e_{23} - e_{45} + e_{67} + e_{123} - e_{145} + e_{167} + e_{234567} - 3e_{1234567}). \quad (7.20)$$

Again, $\mathrm{Spec}(f_{-3}) = -\frac{3}{4} = \frac{k}{d}$ and so f_{-3} is another exceptional root of -1.

As expected, we can also see that the roots f and ω are related via the reversion whereas $f_3 = -\bar{f}_{-3}$, $f_2 = -\bar{f}_{-2}$, $f_1 = -\bar{f}_{-1}$ where $\bar{\ }$ denotes the conjugation in $C\ell(7,0)$.

8. Conclusions

We proved that in all cases $\mathrm{Scal}(f) = 0$ for every square root of -1 in A isomorphic to $C\ell(p,q)$. We distinguished ordinary square roots of -1, and exceptional ones.

In all cases the ordinary square roots f of -1 constitute a unique conjugacy class of dimension $\dim(A)/2$ which has as many connected components as the group $G(A)$ of invertible elements in A. Furthermore, we have $\mathrm{Spec}(f) = 0$ (zero pseudoscalar part) if the associated ring is $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$. The exceptional square roots of -1 only exist if $A \cong \mathcal{M}(2d,\mathbb{C})$ (see Section 7).

For $A = \mathcal{M}(2d,\mathbb{R})$ of Section 3, the centralizer and the conjugacy class of a square root f of -1 both have $\mathbb{R}$-dimension $2d^2$ with two connected components, pictured in Fig. 2 for $d = 1$.

For $A = \mathcal{M}(2d,\mathbb{R}^2) = \mathcal{M}(2d,\mathbb{R}) \times \mathcal{M}(2d,\mathbb{R})$ of Section 4, the square roots of $(-1,-1)$ are pairs of two square roots of -1 in $\mathcal{M}(2d,\mathbb{R})$. They constitute a unique conjugacy class with four connected components, each of dimension $4d^2$. Regarding the four connected components, the group $\mathrm{Inn}(A)$ induces the permutations of the Klein group whereas the quotient group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ is isomorphic to the group of isometries of a Euclidean square in 2D.

For A = M (d; H) of Section 5, the submanifold of the square roots f of 1 is a single connected conjugacy class of R-dimension 2 d^2 equal to the R-dimension of the centralizer of every f . The easiest example is H itself for d = 1.

For A = M (d; H^2) = M (2d; H) M (2d; H) of Section 6, the square roots of (1; 1) are pairs of two square roots (f; f ^{0}) of 1 in M (2d; H) and constitute a unique connected conjugacy class of R-dimension 4 d^2. The group Aut(A) has two connected components: the neutral component Inn(A) connected to the identity and the second component containing the swap automorphism (f; f ^{0}) 7! (f ^{0}; f). The simplest case for d = 1 is H^2 isomorphic to C`(0; 3).

For A = M (2d; C) of Section 7, the square roots of 1 are in bijection to the idempotents. First, the ordinary square roots of 1 (with k = 0) constitute a conjugacy class of R-dimension 4 d^2 of a single connected component which is invariant under Aut(A). Second, there are 2d conjugacy classes of exceptional square roots of 1, each composed of a single connected component, characterized by equality Spec(f) = k=d (the pseudoscalar coe cient) with k 2 f 1; 2; : : : ; dg, and their R-dimensions are 4(d^2 k^2). The group Aut(A) includes conjugation of the pseudoscalar ! 7! ! which maps the conjugacy class associated with k to the class associated with k. The simplest case for d = 1 is the Pauli matrix algebra isomorphic to the geometric algebra C`(3; 0) of 3D Euclidean space R^3, and to complex qu;on he1swa1(i)-11.3031.52 0 Td [(A)3.499e sw2-11.301(e1

k	f_k	$\Delta_k(t)$
1	$\omega = e_{123}$	$(t-i)^2$
0	e_{23}	$(t-i)(t+i)$
−1	$-\omega = -e_{123}$	$(t+i)^2$

Table 1. Square roots of -1 in $C\ell(3,0) = M(2,\mathbb{C})$, $d = 1$

k	f_k	$\Delta_k(t)$
2	$\omega = e_{12345}$	$(t-i)^4$
1	$\frac{1}{2}(e_{23} + e_{123} - e_{2345} + e_{12345})$	$(t-i)^3(t+i)$
0	e_{123}	$(t-i)^2(t+i)^2$
−1	$\frac{1}{2}(e_{23} + e_{123} + e_{2345} - e_{12345})$	$(t-i)(t+i)^3$
−2	$-\omega = -e_{12345}$	$(t+i)^4$

Table 2. Square roots of -1 in $C\ell(4,1) = M(4,\mathbb{C})$, $d = 2$

k	f_k	$\Delta_k(t)$
2	$\omega = e_{12345}$	$(t-i)^4$
1	$\frac{1}{2}(e_{3} + e_{12} + e_{45} + e_{12345})$	$(t-i)^3(t+i)$
0	e_{45}	$(t-i)^2(t+i)^2$
−1	$\frac{1}{2}(-e_{3} + e_{12} + e_{45} - e_{12345})$	$(t-i)(t+i)^3$
−2	$-\omega = -e_{12345}$	$(t+i)^4$

Table 3. Square roots of -1 in $C\ell(0,5) = M(4,\mathbb{C})$, $d = 2$

k	f_k	$\Delta_k(t)$
2	$\omega = e_{12345}$	$(t-i)^4$
1	$\frac{1}{2}(e_{3} + e_{134} + e_{235} + \omega)$	$(t-i)^3(t+i)$
0	e_{134}	$(t-i)^2(t+i)^2$
−1	$\frac{1}{2}(-e_{3} + e_{134} + e_{235} - \omega)$	$(t-i)(t+i)^3$
−2	$-\omega = -e_{12345}$	$(t+i)^4$

Table 4. Square roots of -1 in $C\ell(2,3) = M(4,\mathbb{C})$, $d = 2$

Appendix B. A sample Maple worksheet

In this appendix we show a computation of roots of -1 in $C\ell(3,0)$ in CLIFFORD. Although these computations certainly can be performed by hand,

the actual Maple worksheets where these computations have been performed, see [18].

>

$$M_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The basis element e is represented by the following matrix

$$M_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The basis element e2 is represented by the following matrix

$$M_3 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The basis element e3 is represented by the following matrix

$$M_4 := \begin{bmatrix} 0 & \mathbf{e23} \\ \mathbf{e23} & 0 \end{bmatrix}$$

The basis element e 2 is represented by the following matrix

$$M_5 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The basis element e 3 is represented by the following matrix

$$M_6 := \begin{bmatrix} 0 & \mathbf{e23} \\ \mathbf{e23} & 0 \end{bmatrix}$$

The basis element e23 is represented by the following matrix

$$M_7 := \begin{bmatrix} \mathbf{e23} & 0 \\ 0 & \mathbf{e23} \end{bmatrix}$$

The basis element e 23 is represented by the following matrix

$$M_8 := \begin{bmatrix} \mathbf{e23} & 0 \\ 0 & \mathbf{e23} \end{bmatrix}$$

```
> f/  , =phi(F/  , M)  °°  element in Cl(3  ) correspondi to F/  ,
  cmul(f/  , f/  ,)  °°  checki that this element is a root of
  Mu/  ,  °°  recalli minpoly of matrix F/  ,
  subs(e23=I evalm(subs(t=evalm(F/  ,) Mu/  ,)))  °°  F/  , in Mu/  ,
  mu/  , =subs(I=reprI Mu/  ,)  °°  defini minpoly of f/  ,
  cmul(f/  , reprI Id)  °°  f/  , satisfies mu/  ,
```

$$f_1 := \mathbf{e123}$$

$$\mathbf{Id};\quad t + I;\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$_1 := t + \mathbf{e123};\quad 0$$

Functions RdimCentralizer and RdimConjugClass of d and k compute the real dimension of the centralizer Cent(f) and the conjugacy class of f (see (7.4)).

```
> RdimCentralizer =(d k) >2 ((d k) 2 (d k) 2)  °°  from the theory
> RdimConju Class =(d k) >4 (d 2 k 2)  °°  from the theory
```

$$\mathbf{RdimCentralizer} := (d; k) \;!\; 2(d+k)^2 + 2(d \quad k)^2$$

$$\mathbf{RdimConjugClass} := (d; k) \;!\; 4d^2 \quad 4k^2$$

Now, we compute the centralizers of the roots and use notation $d; k; n_1; n_2$ displayed in Examples.

Case $k = 1$:

```
> d =, k =, n, =d k n2 =d k
  A, =di (I n  I n2)  °°  this is the first matrix root of
```

$$\mathbf{n1} := 2;\quad \mathbf{n2} := 0;\quad \mathbf{A1} := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

```
> f/ e122=phi(A ; M)  cmul(f/  , f/  ,)  Scal(f/  ,)  3( ) 2 ( ) 2 (  hh34a )  .l(f 3  4r3 2 hTd(
```

```
> f  =phi(A  M)  cmul(f   f  )  Scal(f  )  Spec(f  )
```

f_0 := **e23**; **Id**; 0; 0

```
> LL =Centralizer(f   clibas)   ::  centralizer of f
  dimCentralizer =nops(LL )   ::  real dimension of centralizer of f
  RdimCentralizer(d k)   ::  dimension of centralizer of f  from theory
  evalb(dimCentralizer=RdimCentralizer(d k))   ::  checking equality
```

LL0 := [**Id**; **e1**; **e23**; **e123**]

dimCentralizer := 4 ; 4; **true**

Case k = 1 :

```
> d =  k =   n  =d k n2 =d k
```

```
> 'F/  .'=evalm(F/  .)  °°  square root of   in C(2)
  Mu/   .   °°  minpoly of matrix F/   .
  'f/  .'=f/    .   °°  square root of   in Cl(3  )
  mu/   .   °°  minpoly of element f/   .
```

$$F_1 = \begin{bmatrix} \mathbf{e23} & 0 \\ 0 & \mathbf{e23} \end{bmatrix};\quad t + I$$

$$f_1 = \mathbf{e123}$$

[13] D. Hestenes, **New Foundations for Classical Mechanics**. Kluwer, Dordrecht, 1999.

[14] D. Hestenes, G. Sobczyk, **Clifford Algebra to Geometric Calculus**. Kluwer, Dordrecht, 1984.

[15] E. Hitzer, **Quaternion Fourier Transform on Quaternion Fields and Generalizations**. Adv. Appl. Clifford Algebras 17(3) (2007), 497{517.

[16] E. Hitzer, **OPS-QFTs: A New Type of Quaternion Fourier Transforms Based on the Orthogonal Planes Split with One or Two General Pure Quaternions**. Numerical Analysis and Applied Mathematics ICNAAM 2011, AIP Conf. Proc. 1389 (2011), 280{283; DOI: 10.1063/1.3636721.

[17] E. Hitzer, R. Ablamowicz, **Geometric Roots of** 1 **in Clifford Algebras** $C\ell(p,q)$ **with** $p+q$ 4. Adv. Appl. Clifford Algebras, Online First, 13 July 2010. DOI: 10.1007/s00006-010-0240-x.

[18] E. Hitzer, J. Helmstetter, and R. Ablamowicz, Maple worksheets created with CLIFFORD for a verification of results presented in this paper, http://math.tntech.edu/rafal/publications.html (c 2012).

[19] E. Hitzer, B. Mawardi, **Uncertainty Principle for the Clifford Algebra** $C\ell_{n,0}$, $n = 3 \pmod 4$ **Based on Clifford Fourier Transform**. In Springer SCI book series