Using Periodicity Theorems for Computations in Higher Dimensional Clifford Algebras

[illegible]

Abstract [illegible]

theorem states that there exists a basis for V such that the quadratic form Q is diagonal with entries $1, 0$ in the real case (0 only when Q is degenerate; just 1's in the non-degenerate complex case). Under these isomorphisms, the real quadratic space (Q, V) with a non-degenerate Q is isomorphic to a space $\mathbb{R}^{p,q}$

(the graded tensor product $\hat{\otimes}$ is defined below and we use equality for categorical isomorphisms). Similarly we get for Clifford algebras

$$C\ell(V_1 + V_2; Q_1 \perp Q_2) = C\ell(V_1; Q_1) \hat{\otimes} C\ell(V_2; Q_2); \tag{8}$$

and it is this decomposition which will be used below to compute in CLIFFORD in dimensions ≤ 9. For Clifford algebras with non-symmetric bilinear forms such a decomposition is in general not direct, see [18].

2.4 Tensor products of (graded) algebras

Let $(A; m_A)$ and $(B; m_B)$ be K

In the Grassmann algebra case, splitting the space $V = V_1 + V_2$ with n basis vectors e_i into two sets with, respectively, p $(1 \le i \le p)$ and q $(p < i \le n)$ vectors, we get the maps $e_i \mapsto e_i \wedge 1$ $(i \le p)$ and $e_j \mapsto 1 \wedge e_j$ $(p < j \le n)$. In the CAS computations below we will *standardize* the indices, that is, we will reindex $j \mapsto j - p$ so that $i \in \{1, \ldots, p\}$ and $j - p \in \{1, \ldots, n - p\}$. The graded tensor product ensures that we still have the desired anti-commutation relations

$(e_i \wedge$

ploys (graded) algebra isomorphisms described on the generators of the factor Clifford algebras inside the ambient Clifford algebra. This leads to the well-known periodicity relations which are summarized in the following[3].

Theorem 2. For real Clifford algebras we have the following periodicity theorems and isomorphisms:

Theorem 3 ([14], Theorem 5.8)**.** With the notation as above, let V_2 have dimension $2k$ and let ω be the volume element in $C\ell(V_2;Q_2)$ with $\omega^2 = \lambda \neq 0$. There exists a vector space isomorphism between the module $C\ell(V_1 \oplus V_2; Q_1 \perp Q_2)$ and the module $C\ell(V_1; \frac{1}{\lambda}Q_1) \otimes C\ell(V_2;Q_2)$ given on generators as $(x;y) \mapsto x \otimes \omega + 1 \otimes y$, and there is a graded algebra isomorphism

$$C\ell(V_1 \oplus V_2; Q_1 \perp Q_2) \simeq C\ell(V_1; \frac{1}{\lambda}Q_1) \otimes C\ell(V_2;Q_2). \tag{17}$$

The involutions extend as $\widehat{(x \otimes y)} \simeq \hat{x} \otimes \hat{y}$ and $\mathrm{rev}(x \otimes y) \simeq \mathrm{rev}(x) \otimes \mathrm{rev}(y)$ if $|x| \equiv 0 \bmod 2$ even and $\mathrm{rev}(x \otimes y) \simeq \mathrm{rev}(x) \otimes \mathrm{rev}(\hat{y})$ otherwise. Then all periodicity isomorphisms in Theorem 2 are special cases of this one.[4]

To exemplify this, let $(x;y)$ be any pair of generators with $x \in V_1$ and $y \in V_2$ which upon the embedding $V_1 \oplus V_2 \hookrightarrow C\ell(V_1 \oplus V_2; Q_1 \perp Q_2)$ we write as the sum $x + y$. Then,

$$(x+y)^2 = x^2 + (xy + yx) + y^2 = (Q_1(x) + Q_2(y))1 = (Q_1 \perp Q_2)(x;y) \tag{18}$$

due to the orthogonality of x and y. On the other hand, in the (ungraded) tensor product algebra in the right-hand-side of (17) we find, as expected,

$$\begin{aligned}
&(x \otimes \omega + 1 \otimes y)^2 \\
&\quad = (x \otimes \omega)(x \otimes \omega) + (x \otimes \omega)(1 \otimes y) + (1 \otimes y)(x \otimes \omega) + (1 \otimes y)(1 \otimes y) \\
&\quad = x^2 \otimes \omega^2 + x \otimes \omega y + x \otimes y\omega + 1 \otimes y^2 \\
&\quad = \frac{1}{\lambda} Q_1(x) 1 \otimes \lambda
\end{aligned}$$

$$(x_1 x_2) \quad w^2 + (x_1 \wedge 1) \quad w y_2 + (1 \wedge x_2) \quad y_1 w + (1 \wedge 1) \quad (y_1 y_2) = (x_1 \wedge x_2) \quad I\ 1 + x_1 \quad w y_2 + x_2 \quad y_1 w + 1 \quad (y_1 \wedge y_2). \qquad (21)$$

The isomorphism in (17) is given by the procedures bas2Tbas (from left to right) and its inverse Tbas2bas (from right to left). In the worksheets [7] we show both procedures as well as we verify the assertions regarding the involutions.

2.6 Spinor representations, Clifford valued matrix representations

A Clifford algebra is an abstract algebra, but we may want to realize it as a concrete matrix algebra. It is, however, well known that matrix representations may be very inefficient for CAS purposes. The simplest representation is the (left) regular representation, sending $a \in A \mapsto \lambda_a = m_A(a, \cdot) \in \mathrm{End}(A)$, the left multiplication operator by a. This representation is usually highly reducible. The smallest faithful representations of a Clifford algebra are given by spinor representations.[5] Algebraically, a spinor representation is given by a minimal left ideal which can be generated by left multiplication from a primitive idempotent $f_i = f_i^2$ with $\nexists f_k, f_l \neq 0$ idempotents such that $f_i = f_k + f_l$ and $f_k f_l = f_l f_k = 0$. The vector space $S_i := C\ell_{p,q} f_i$ is a spinor space, and it carries a faithful irreducible representation of $C\ell_{p,q}$ for simple algebras.[6] However, when $C\ell_{p}$

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S = C ; f = n_Rf f = f = –(+

underlying a Clifford algebra using the Grassmann $\mathbb{Z}_2$-grading.[9] That is, mapping the generators $e_i \mapsto -e_i$ in both cases. From the property of the Grassmann functor (7), by replacing (formally) $\hat{}\,V \mapsto -V$ under the grade involution $\hat{}$, we derive

$$\bigwedge(-V \oplus -W) = \bigwedge(-V)\,\hat{\otimes}\,\bigwedge(-W).$$

This amounts to saying that $\hat{}\,|_{V+W} = \hat{}\,|_V \otimes \hat{}\,|_W$, and the same is true for the Clifford functor $C\ell$. The grade involution on graded and ungraded tensor products of Clifford algebras reads then:

$$\hat{} : C\ell_{p,q}\ \hat{\otimes}\ C\ell$$

3 Computing with CLIFFORD and Bigebra in tensor algebras

... M ... cod ... o ... n o ... od c ... o ... o d ... n CLIFFORD nd Bigebra ... o ... c ... nd ... c ... , ... M ... o ... cod ... o ... d ... c ... d ... od ... o ... d ...

Lo d ng ... c ... ng >with(Clifford);with(Bigebra); ... o ... o ... d nc on o ... o d ... y $C\ell$; on ... n ... d ... od ... n ... d ... n on >dim_V:=2+2; nd ... n ... o >B:=linalg[diag](1$2,-1$2); B ... n e_i ... n ... ng ... e1we4 . nd o e e ; c ... Id . nd o ... d ... n ... y o ... o d ... Bigebra ... o ... o ... d ... d ... n o od c &t ... c ... n ... nd oc ... n ... n o od c $e\otimes e$ d &t(e1we2,e1) nd ... on o ... n o ... n ... d y >switch(&t(e1,e2),1) = &t(e2,e1) ... ng d ... d ... c ... o ... n ... d ... d c ... >gswitch(&t(e1,e2),1) = -&t(e2,e1) ... d ... d ... c ... nd ... i ... n ... c ... d 1 ... n ... c ... [g]switch . o ... nd ... n A ... n yo ... y y ng >?switch nd >?gswitch ... M ... o .

... o d od c cmul y d ... c y ... n ... o B ... n o ... >cmul(e1,e2)=e1we2+B[1,2]*Id o ... c n o B o ny o ... M ... n ... c y n o ... on ... n ... >cmul[K](e1,e2) = e1we2+K[1,2]*Id o ng . o co ... n d ... n ... o d ... n ... o ...

L ... B B B ... o d ... n ... o o $C\ell_{p+r,q+s}$; $C\ell_{p,q}$ nd $C\ell_{r,s}$ nd ... bas2GTbas ... d ... d ... o o ... n y $e_I \in C\ell_{p,q}$ &t(eI,Id) $I \in \{ ;:::; p+q\}$ nd $e_J \in C\ell_{r,s}$ &t(Id,eJ) $J \in \{ ;:::r+s\}$ ... n ... oc ... d ... cmulGTensor ... n ... o d ... od c ... n ... graded ... n o od c o ... o d ... n ... o ... n ... d ... n ... c ... on ,

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... d ... n ... o ... n ... o ... n CLIFFORD nd Bigebra ... B ... o ... o ... n ... c n o ... d ... o ... n ... n ... o B ... nd ... n ... d ... n ... n ... d co ... on ... o ... d n ... o d ... $C\ell(B)$ o n ... y ... n ... o B ... , ... , ... , ... , ... n

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cmulGTensor:=
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$$C\ell_{p+k,q+k} \simeq C\ell_{p,q} \otimes \underbrace{C\ell_{1,1} \otimes \cdots \otimes C\ell_{1,1}}_{k \text{ factors}}, \tag{33}$$

or use the mod 8 periodicity.

4 Computations using matrix algebras over Clifford numbers

The isomorphism 6) from Theorem 2 was explicitly defined by Lounesto in [21, Sect. 16.3]. We will use this matrix approach to perform computations in $C\ell_{8,2} \simeq \mathrm{Mat}(2, C\ell_{7,1})$ [11]. Let $\{e_1, \ldots, e_8\}$ be an orthonormal basis of $\mathbb{R}^{7,1}$ generating the Clifford algebra $C\ell_{7,1}$ such that $e_i^2 = 1$ for $1 \leq i \leq 7$, $e_8^2 = -1$, and $e_i e_j = -e_j e_i$ for $i, j \leq 8$ and $i \neq j$. The following 2×2 matrices (compare with (23))

$$E_i = \begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, 8; \quad E_9 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad E_{10} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{34}$$

anti-commute and generate $C\ell_{8,2}$.[13] In order to effectively compute in $C\ell_{8,2}$

The Mat$(2; C\ell_{p,q})$ case is different. Due to the choice of a spinor basis for $C\ell_{1,1}$, the grade involution depends on this choice. Using the basis as used in Section 2.6 equation (23), we code the graded involution as

Listing 3 Mat$(2; C\ell_{p,q})$ main involution

```
# Mgradeinv : grade involution on Mat(2,CL_p,q)
Mgradeinv:= proc(x)
   linalg[matrix](2,2, [    gradeinv(x[1,1]),-    gradeinv(x[1,2]),
                          - gradeinv(x[2,1]),     gradeinv(x[2,2])]);
end proc:
```

This reflects the fact that in this spinor basis the non zero diagonal terms (e_i) of generators are odd, while the non zero off diagonal terms are even (± 1) and need an additional minus sign.

The reversion is more complicated as it involves swapping of generators between the two factors of the product representations or involves the chosen spinor basis. The graded tensor case just needs an additional sign due to the swapping of the two factors of the product:

Listing 4 Graded reversion

```
# GTreversion : reversion involution on CL_p,q (x) CL_r,s
#                   !! works for general bilinear forms B1 & B2 !!
GTreversion:= proc(x,B1,B2)    local f2;
    f2:=(a,b)->&t(     reversion(b,B1),   reversion(a,B2)); # note order
                                                             # of a,b
    eval( subs(`&t`=f2,    gswitchg
```

The reversion in the $\mathrm{Mat}(2;C\ell_{p,q})$ case depends on the basis chosen in (23). It swaps the diagonal entries and has to apply the grade involution to the second column.[18]

Listing 6 $\mathrm{Mat}(2;C\ell_{p,q})$ reversion

```
#   Mreversion :  reversion on Mat(2,CL_p,q)
#            NOTE: depends on spinor basis for CL_1,1
Mreversion:= proc(x,B)
  linalg[matrix](2,2,
   [ gradeinv( reversion(x[2,2],B)),      gradeinv( reversion(x[1,2],B)),
     gradeinv( reversion(x[2,1],B)),      gradeinv( reversion(x[1,1],B))]);
end proc:
```

References

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