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Affect, Behavioral Schemas, and the Proving Process*

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In this largely theoretical paper, we discuss the relation between a kind of affect, behavioral schemas, and aspects of the proving process. We begin with affect as described in the mathematics education

tutor then needed to take an action in response, as even doing nothing would likely have been interpreted by Sofia as an action or as having meaning. But to act usefully the tutor needed to understand why Sofia made her peculiar suggestion. We believe her suggestion was triggered by some kind of affect, but a kind not among the three major aspects previously discussed. On viewing the video, we found no evidence of an emotional response, and we have no reason to believe that Sofia had an underlying belief or attitude related to this or other similar incidents. We believe Sofia's suggestion was triggered by a feeling of confusion or by a feeling of not knowing what to do next.

The above example of Sofia suggests that it would be useful to expand one's view of affect somewhat, and that particular aspects of affect may be linked to specific parts of the proving process. DeBellis and Goldin [2–4] have added a forth main aspect to affect, namely, *values*. While this addition is a reasonable extension of affect, it also does not account for Sofia's behavior. DeBellis and Goldin further point out that 'affect is not auxiliary to cognition; it is centrally intertwined with it' [5]. They see 'affect as a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition' [3, p. 2-249]. We now add a fifth aspect to affect, namely, *feelings*.

2.2 Feelings

DeBellis and Goldin, as well as Ortony, Clore, and Collins [8], see feelings as having an appraisal value that can be either positive or negative. In addition, Clore has considered 'feelings-as-information'. While DeBellis and Goldin [4] refer to 'emotional feelings,' Damasio [9] clearly distinguishes between emotions and feelings, with the former being public and the latter being private (p. 27). 'Emotions play out in the theatre of the body. Feelings play out in the theatre of the mind' (p. 28). Most of the emotions that Damasio considers, such as joy and sorrow, as well as some less intense emotions, are a complex collection of chemical and neural responses to a stimulus that may produce bodily changes, such as changes in one's heart rate, temperature, and so forth (p. 53).

Because this distinction between feelings and emotions may be somewhat counterintuitive, we paraphrase one of Damasio's salient examples. As doctors were placing tiny electrodes in the mesencephalon of the brain stem of a 65-year old woman suffering from Parkinson's disease, the patient abruptly stopped her ongoing conversation, began to look sad (evidence of emotion), and a *few seconds later* suddenly began to cry. She then said she had no energy to go on living (evidence of a feeling). The doctors quickly removed the offending electrode and the sobbing stopped as abruptly as it had begun, and the sadness vanished from the woman's face.

> The sequence of events in this patient reveals that the emotion sadness came first. The feeling of sadness followed … Once the stimulation ceased these manifestations waned and then vanished. The emotion disappeared and so did the feeling. … The importance of this rare neurological incident is

introspection which came first. This woman's case helps us see through the conflation [between emotions and feelings]' [8, p. 57].

experiences are more peripheral, vague, and of lower resolution. If new sensory experiences are

4.1 An illustration of a behavioral schema

Mary, an advanced graduate student in mathematics described to us a <situation, action> pair in proving that a compact subset A of R^n is bounded. Mary and two fellow graduate student assumed A to be unbounded and were able to construct an open cover of A that had no finite subcover. They immediately observed, without further reflection, that this contradicted the compactness of A and that this proved the result. Mary, who had never studied formal logic, reported that, upon finding the cover had no finite subcover, she immediately knew the result was proved. She did not reflect on the logical structure of what had transpired in an effort to explicitly warrant that the proof was complete. The other two students also did not appear to require reflection or an explicit warrant.

We infer that each student had recognized the situation as similar to one that they had experienced many times previously involving a hypothesis, a conclusion assumed false, and a resulting contradiction. The mental action was simply deciding the result had been proved. The link between the situation and the mental action appeared to be automatic and not to require reflection or a warrant. We see this as due to the students' extensive proof constructing experiences. However, many less experienced students require considerable reflection and wonder needlessly what should be contradicted. We see such <situation, action> pairs as common and as playing a significant role in the proving process.

4.2 The genesis and enactment of behavioral schemas

We regard <situation, action> pairs that occur regularly as the enactment of enduring mental structures that we call *behavioral schemas*. These are partly procedural knowledge, that is, knowing *how* to act, and are similar to what Mason and Spence [16] have called 'knowing *to* act in the moment'. We view behavioral schemas as belonging to a person's knowledge base. The action produced by the enactment of a behavioral schema might be simple, as in the above example, or complex, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioral schema. When viewed in a large grain-size, behavioral schemas might also be regarded as habits of mind [14]. Habits of mind are similar to physical habits, but people are even less likely to be aware of them.

It appears that consciousness plays an essent

(3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.

(4) A behavioral schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioral schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioral schema. That is, one cannot ‘chain together’ behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.

(5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person’s actions, but such influence is normally not precise enough for doing mathematics.

(6) Behavioral schemas are acquired (learned) through habituation. That is, to acquire such a schema a person should carry out the appropriate action correctly a number of times. Changing an detrimental behavioral schema requires similar, perhaps longer, practice.

Elaborations of, and justifications for, this theoretical sketch can be found in [15].

5. The course

The setting from which our data are taken is a design experiment, consisting of a Modified Moore Method course [18,19], whose sole purpose is to improve the proving skills of beginning graduate and advanced undergraduate mathematics students. The course is consistent with a constructivist point of view, in that we attempt to maximize students’ proof writing experiences. It is also somewhat Vygotskian in that we represent to the students how the mathematics community writes proofs.

The students are given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students present their proofs in class, and the proofs are critiqued. Suggestions for improvements in their notation and style of writing are also given. There are no formal lectures, and all comments and conversations are based on students’ work. The course carries three credits and lasts one semester. It meets for one hour and fifteen minutes twice a week, making 30 class meetings per semester. We have now taught three, of a projected eight iterations of the course. There are two versions of the course, and either or both can be taken for credit. One version covers some basic ideas about sets, functions, real analysis, and semigroups. The other version covers sets, functions, some real analysis, and topology. The specific topics covered are of less importance than giving students opportunities to experience as many different types of proofs as possible.

6. Feelings and the proving process

6.1 Nonemotional cognitive feelings and

events that had taken place two years earlier, when Mary was taking both a pilot version of our proofs course (section 5 above) and Dr. K's real analysis course. In the homework for Dr. K's course, Mary needed to prove many statements that included phrases like 'For all real numbers *x*,' where *x* represented a variable (the situation). In her proofs, Mary needed to write something like 'Let *x* be a number,' where *x* represented an arbitrary, but fixed number (the action). Dr. K often discussed Mary's proofs with her, and in particular, thought she carried out this action based on his authority.

When Mary was interviewed about this <situation, action> pair she said the following:

Mary: At that point [early in Dr. K's real analysis course] my biggest idea was, well he said to 'do it', so I'm going to do it because I want to get full credit. And so I didn't have a real sense of why it worked.

Int: Did you have any feeling … if it was positive or negative, or extra …

Mary: Well, I guess I had a feeling of discomfort …

Int: Did this particular feature [having to fix *x*] keep coming up in proofs?

Mary: … it comes up a lot and what happened, and I don't remember [exactly] when, is that instead of being rote and kind of uncomfortable, it started to just make sense … By the end of the semester this was very comfortable for me.

We infer Mary developed both the behavioral schema and the associated feeling of appropriateness only after executing the <situation, action> pair numerous times. In early executions of this <situation, action> pair, Mary carried it out mainly based on Dr. K's authority. In addition, after completing each such proof, Mary attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof. Only after repeatedly executing this <situation, action> pair, and convincing herself that the individual proofs were correct, did she develop a feeling of appropriateness.

6.3 Students who focus too soon on the hypotheses

In a paper reporting on a mid-level undergraduate transition-to-proof course, Moore [20] described students' attempts to prove: *Let f and g be functions on A. If $f \circ g$ is one-to-one, then g is one-to-one.* This was a final examination question and 'all but one student [of 16] incorrectly attempted to begin [the problem-centered part of] the proof with the hypothesis – $f \circ g$ is one-to-one – rather than [starting to prove the conclusion] by assuming

Theorem 29: *Let X and Y be topological spaces and $f: X \to Y$ be a homeomorphism of X onto Y. If X is a Hausdorff space, then so is Y.* Because only ten minutes of class time remained, we asked Willy to 'do the set-up', that is, present the formal-rhetorical part of the proof. Willy had indicated that he had not yet proved the theorem. However, early on in the course, he had developed some ability to write the formal-rhetorical parts of simple proofs.

On the left side of the board, Willy wrote:

Proof. Let X and Y be topological spaces.

Let $f: X \to Y$ be a homeomorphism of X onto Y.

Suppose X is a Hausdorff space.

. . .

Then Y is a Hausdorff space.

Then, on the right side of the board, he listed:

homeomorphism

one-to-one

onto

Writing the formal-rhetorical part of a proof exposes the 'real problem' in this theorem, something Willy might have found tractable. Indeed, by the next class meeting, he had constructed a proof in the way we had expected.

6.4 *Sofia's reaction to not having an idea*

In section 6.2, we reported an incident of access to a rare first-person account that illustrated Mary's joint development of a feeling of appropriateness and a behavioral schema. In contrast, in this section, we describe how *we* used our theoretical perspective about feelings and behavioral schemas *to infer* that Sophia's progress was blocked by multiple enactments of an 'unreflective guess' schema, and how our intervention appears to have weakened that schema and improved her

7. Conclusion

In this paper we have described a design experiment to develop a course whose sole purpose is to improve advanced mathematics students' proving skills. The course appears to be useful because many advanced students have difficulty constructing proofs, and students' proofs are used as a major component in assessments of their understanding of content courses, such as abstract algebra or real analysis.

We have discussed the nature of feelings, especially nonemotional cognitive feelings and treated them as a part of affect. Also, a theoretical framework for analysing student progress with proofs is beginning to emerge. For example, we have suggested it is useful to distinguish between the formal-rhetorical parts of proofs and the problem-centered parts.

We further introduced a theoretical perspective suggesting that much of the proving process depends on procedural knowledge in the form of small habits of mind or behavioral schemas, some of which are beneficial, while others tend to produce difficulties. The way feelings can both arise from, and contribute to, the enac

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