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GEOMETRIC ROOTS OF -1 IN CLIFFORD ALGEBRAS $C_{p,q}$ WITH $p+q \leq 4$

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Geometric Roots of -1 in Clifford Algebras $C\ell_{p,q}$ with $p+q \leq 4$

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Abstract. It is known that Clifford (geometric) algebra offers a geometric interpretation for square roots of -1 in the form of blades that square to minus 1. This extends to a geometric interpretation of quaternions as the side face bivectors of a unit cube. Research has been done [1] on the biquaternion roots of -1, abandoning the restriction to blades. Biquaternions are isomorphic to the Clifford (geometric) algebra $C\ell_3$ of $\mathbb{R}^3$. All these roots of -1 find immediate applications in the construction of new types of geometric Clifford Fourier transformations.

We now extend this research to general algebras $C\ell_{p,q}$. We fully derive the geometric roots of -1 for the Clifford (geometric) algebras with $p+q \leq 4$.

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1. Introduction

The British mathematician W.K. Clifford created his *geometric algebras*[1] in 1878 inspired by the works of Hamilton on quaternions and by Grassmann's exterior algebra. Grassmann invented the antisymmetric outer product of vectors, that regards the oriented parallelogram area spanned by two vectors as a new type of number, commonly called bivector. The bivector represents its own plane, because outer products with vectors in the plane vanish. In three dimensions the outer

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[1] In his original publication [2] Clifford first used the term *geometric algebras*. Subsequently in mathematics the new term *Clifford algebras*

product of three linearly independent vectors defines a so-called trivector with the magnitude of the volume of the parallelepiped spanned by the vectors. Its orientation (sign) depends on the handedness of the three vectors.

In the Clifford algebra [16] of $\mathbb{R}^3$ the three bivector side faces of a unit cube $\{e_1e_2, e_2e_3, e_3e_1\}$ oriented along the three coordinate directions $\{e_1, e_2, e_3\}$ correspond to the three quaternion units $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. Like quaternions, these three bivectors square to minus one and generate the rotations in their respective planes.

Beyond that Clifford algebra allows to extend complex numbers to higher dimensions [3, 4] and systematically generalize our knowledge of complex numbers, holomorphic functions and quaternions. It has found rich applications in symbolic computation, physics, robotics, computer graphics, etc. [5, 14, 15, 18]. Since bivectors and trivectors in the Clifford algebras of Euclidean vector spaces square to minus one, we can use them to create new geometric kernels for Fourier transformations. This leads to a large variety of new Fourier transformations, which all deserve to be studied in their own right [5–13, 26, 28, 29].

We will treat both Euclidean (positive definite metric) and non-Euclidean (indefinite metric) vector spaces. We know from Einstein's special theory of relativity that non-Euclidean vector spaces are of fundamental importance in nature [17]. Therefore this paper is about finding square roots of -1 in a non-degenerate Clifford algebra $C\ell_{p,q}$.

2. Clifford (geometric) algebras

The associative geometric product of two vectors $a, b \in \mathbb{R}^{p,q}$, $p + q = n$ is defined as the sum of their symmetric inner product (scalar) and their antisymmetric outer product (bivector)

$$ab = a \cdot b + a \wedge b. \tag{1}$$

We define [20] a real Clifford algebra $C\ell_{p,q}$ as the linear space of all elements generated by the associative (and distributive) bilinear geometric product of vectors of an inner product vector space $\mathbb{R}^{p,q}$, $p + q = n$ over the field of reals $\mathbb{R}$. A Clifford algebra includes the field of reals $\mathbb{R}$ and the vector space $\mathbb{R}^{p,q}$ as grade zero and grade one elements, respectively.

Clifford algebras in one, two and three dimensions have the following basis blades of grade 0 (scalars), grade 1 (vectors), grade 2 (bivectors) and grade 3 (trivectors)

$$\{1, e_1, e_2, e_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{123}\}, \tag{2}$$

where we use abbreviations $\mathbf{e}_{12} = e_1 e_2, \mathbf{e}_{23} = e_2 e_3, \mathbf{e}_{31} = e_3 e_1, \mathbf{e}_{123} = e_1 e_2 e_3$. Every multivector can be expanded in terms of these basis blades with real coefficients. We give examples for $M \in C\ell_{p,q}$, $n = p + q = 1, 2, 3$:

$$M = \alpha + \beta e_1, \tag{3}$$

$$M = \alpha + b_1 e_1 + b_2 e_2 + \beta \mathbf{e}_{12}, \tag{4}$$

$$M = \alpha + b_1 e_1 + b_2 e_2 + b_3 e_3 + c_1 \mathbf{e}_{23} + c_2 \mathbf{e}_{31} + c_3 \mathbf{e}_{12} + \beta \mathbf{e}_{123}. \tag{5}$$

The general notation for the quadratic form of basis vectors in $\mathbb{R}^{p,q}$ is:

$$e_k^2 = \varepsilon_k = \begin{cases} +1 & \text{for } 1 \leq k \leq p, \\ -1 & \text{for } p+1 \leq k \leq p+q = n \end{cases} . \tag{6}$$

We therefore always have $e_k^4 = \varepsilon_k^2 = 1$, and we abbreviate $C\ell_p = C\ell_{p,0}$. We follow the convention that inner and outer products have priority over the geometric product, which saves writing a number of brackets. Therefore, $a \cdot bc$ equals $(a \cdot b)c$ and not $a \cdot (bc)$, etc.

We will frequently use the following basic formulas of Clifford algebra in the rest of this work. The symmetric part of the geometric produly use the following boulyha

Two blades $A_r, B_s \in Cl_{p,q}$ are called orthogonal iff their inner product is zero

$$A_r \perp B_s \iff A_r \cdot B_s = 0. \tag{13}$$

With (11) follows that for $r \leq s$

$$A_r \perp B_s \iff A_r \wedge (B_s I_n) = 0 \iff A_r \wedge \tilde{B}_s = 0, \tag{14}$$

where $\tilde{B}_s = B_s I_n^{-1}$ is the *dual* of B_s, with $I_n^{-1} = \pm I_n$. Likewise (12) shows that for $r + s \leq n, \;\; r, s > 0$

$$A_r \perp \tilde{B}_s \iff A_r \wedge B_s = 0. \tag{15}$$

Example 1. *Let $b, \underline{c} \in Cl_{p,q}, p + q = 3$ be a vector b and a bivector $\underline{c}$ with vanishing outer product. Then by* (15) *the dual vector $c = \tilde{\underline{c}}$ is always perpendicular to b independent of the signature of the underlying vector space $\mathbb{R}^{p,q}, p + q = 3$,*

$$b \wedge \underline{c} = 0 \iff b \cdot c = 0 \iff b \perp c. \tag{16}$$

3. Geometric multivector square roots of -1

Definition 3.1 (Geometric root of -1). A geometric multivector square root (geometric root) of -1 is a multivector $A \in Cl_{p,q}$ with

$$A^2 = AA = -1. \tag{17}$$

An immediate application of this definition is the generalization of the famous Euler formula to geometric roots A

$$e^{\varphi A} = \cos\phi + A \sin\phi. \tag{18}$$

For example, Lounesto considers e.g. $\cos\phi + e_{12}\sin\phi$ in Cl_2 in [20] on page 29.

Theorem 3.2. *Every multivector square root A of -1 is subject to $n + 1 = p + q + 1$ grade-wise constraints:*

$$A^2 = \langle AA \rangle = -1, \tag{19}$$

and

$$\langle AA \rangle_k = 0, \qquad 1 \leq k \leq n, \tag{20}$$

where $\langle AA \rangle_k$ denotes the k-th vector part of AA, and $\langle AA \rangle = \langle AA \rangle_0$.

We point out that $\langle AA \rangle$ is identical to the scalar product $A * A$ of [3]. In the following we call the scalar equation (19) the *root equation* of $Cl_{p,q}$ and (20) the *constraints*. Depending on the value of k, each k-vector constraint represents $\binom{n}{k}$ scalar equations. We will sometimes conveniently split up a k-vector constraint equation and still call the resulting partial equations *constraints*.

4. Case $n = 1$

We have two algebras $C\ell_{1}$ and $C\ell_{0,1}$. There is only one basis vector e_1 with square $e_1^2 = \varepsilon_1$. The two Clifford algebras are two dimensional with general elements (multivectors)

$$\alpha + \beta e_1, \qquad \alpha, \beta \in \mathbb{R}. \tag{21}$$

The square of such a multivector is

$$(\alpha + \beta e_1)^2 = \alpha^2 + \varepsilon_1 \beta^2 + 2\alpha\beta e_1 = -1, \tag{22}$$

which has the scalar part (root equation)

$$\alpha^2 + \varepsilon_1 \beta^2 = -1, \tag{23}$$

and the vector part (constraint)

$$2\alpha\beta e_1 = 0. \tag{24}$$

We see that the left hand side of (23) is always greater or equal to zero if $\varepsilon_1 = +1$.

and the bivector part

$$2\alpha\beta e_{12} = 0. \tag{33}$$

5.1. Case $n = 2$, $\alpha = 0$

Equations (32) and (33) are now always fulfilled by any b and β. From (31) it follows that

$$b^2 - \beta^2\varepsilon_1\varepsilon_2 = b_1^2\varepsilon_1 + b_2^2\varepsilon_2 - \beta^2\varepsilon_1\varepsilon_2 = -1. \tag{34}$$

Multiplying each side of (24) by $\varepsilon_1\varepsilon_2$ gives the following *root equation*:

$$\beta^2 = b_1^2\varepsilon_2 + b_2^2\varepsilon_1 + \varepsilon_1\varepsilon_2 = \begin{cases} b_1^2 + b_2^2 + 1 & \text{for } C\ell_{2}, \\ -b_1^2 + b_2^2 - 1 & \text{for } C\ell_{1,1}, \\ -b_1^2 - b_2^2 + 1 & \text{for } C\ell_{0,2}. \end{cases} \tag{35}$$

In $C\ell_2$ this includes, for $b_1 = b_2 = 0$, the solution $A = \pm e_{12}$, which also appears in [20] on page 29.

5.2. Case $n = 2$, $\alpha \neq 0$

If $\alpha \neq 0$, then, according to (32) and (33), we have

$$b = 0 \quad \text{and} \quad \beta = 0. \tag{36}$$

Inserting this in (31) gives

$$\alpha^2 = -1, \qquad \alpha \in \mathbb{R} \setminus \{0\}, \tag{37}$$

which has no solution. Therefore, the root equation (35) describes already all possible solutions.

6. Case $n = 3$

We have four algebras $C\ell_3$, $C\ell_{2,1}$, $C\ell_{1,2}$, and $C\ell_{0,3}$ with a non-trivial center spanned by the identity element 1 and the unit pseudoscalar e_{123}. There are three basis vectors e_k, $k \in \{1, 2, 3\}$, with squares $e_k^2 = \varepsilon_k$. The four Clifford algebras are eight dimensional with general elements

$$\alpha + b + \underline{c} + \beta e_{123}, \quad \alpha, \beta \in \mathbb{R}, \quad b = b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{R}^{p,q},\ p + q = 3, \tag{38}$$

with

$$\underline{c} = c_1e_{23} + c_2e_{31} + c_3e_{12} \in \bigwedge^2 \mathbb{R}^{p,q}, \quad c_1, c_2, c_3 \in \mathbb{R}. \tag{39}$$

Setting the square of such a multivector to -1 gives

$(\alpha + b + c$

Grade-wise this results in the following set of constraints: For the scalar part (root equation)

$$\alpha^2 + b^2 + \underline{c}^2 - \beta^2 \varepsilon_1 \varepsilon_2 \varepsilon_3 = -1, \qquad (41)$$

7. Case $n = 4$

We have five central algebras $C\ell_{4}$, $C\ell_{3,1}$, $C\ell_{2,2}$, $C\ell_{1,3}$, and $C\ell_{0,4}$. There are four basis vectors e_k, $k \in \{1, 2, 3, 4\}$ with square $e_k^2 = \varepsilon_k$, $e_k^4 = \varepsilon_k^2 = 1$, $\mathbf{e}_{123}^2 = \mathbf{e}_{123}^{-2}$, and $\mathbf{e}_{123}^4 = 1$. The five Clifford algebras are 16 dimensional with general elements

$$\alpha + b + c^{2}$$

Multiplying out (70) gives

$$\alpha^2 + b^2 + \underline{c}^2 + \beta^2 e_{123}^2 + \varepsilon_4 \alpha'^2 - \varepsilon_4 b'^2 + \varepsilon_4 \underline{c}'^2 - \varepsilon_4 \beta'^2 e_{123}^2$$
$$+ 2\alpha b + 2\alpha \underline{c} + 2\alpha\beta$$

and the trivector part

$$\alpha\beta \; e_{123} + b \; \wedge \underline{c} = 0. \tag{82}$$

Apart from the actual root equation (73) we have therefore the following set of seven constraint equations

$$\underline{c} \cdot \underline{c} \; = -\alpha\alpha \,, \tag{83}$$

$$\alpha b = -\varepsilon_4 b \; \cdot \underline{c} \; - \beta\underline{c}\, e_{123}, \tag{84}$$

$$\alpha b \; = -b \cdot \underline{c} \; - \beta \; \underline{c}\, e_{123}, \tag{85}$$

$$\alpha\underline{c} \; + \alpha \; \underline{c} = b \; \wedge b, \tag{86}$$

$$\alpha\underline{c} + \varepsilon_4\alpha \; \underline{c} \; = (\varepsilon_4\beta \; b \; - \beta b)\, e_{123}, \tag{87}$$

$$-b \wedge \underline{c} = -\underline{c} \wedge b = \alpha\beta\, e_{123}, \tag{88}$$

$$-b \; \wedge \underline{c} = -\underline{c} \wedge b \; = \alpha\beta \; e_{123}. \tag{89}$$

The outer products of (86) with b and b give the following useful identities

$$\alpha b \wedge \underline{c} \; + \alpha \; b \wedge \underline{c} = 0 \overset{(88)}{\Longrightarrow} \alpha \, b \wedge \underline{c} \; = \alpha\alpha \, \beta \, e_{123}, \tag{90}$$

$$\alpha b \; \wedge \underline{c} \; + \alpha \; b \; \wedge \underline{c} = 0 \overset{(89)}{\Longrightarrow} \alpha \, b \; \wedge \underline{c} \; = \alpha\alpha \, \beta \; e_{123}. \tag{91}$$

The inner products (left contractions) of (84) with b and of (85) with b lead to

$$\alpha b \cdot b \; = -\varepsilon_4 \underbrace{b \; \cdot (b \; \cdot \underline{c} \;)}_{0} - \beta \underbrace{b \; \cdot (\underline{c}\, e_{123})}_{(\vec{b}' \;\; \underline{c})\, \mathbf{e}_{123}} = \beta(-b \; \wedge \underline{c})\, e_{123} \overset{(89)}{=} \alpha\beta\beta \; e_{123}^2, \tag{92}$$

$$\alpha b \cdot b \; = -\underbrace{b \cdot (b \cdot \underline{c} \;)}_{0} - \beta \; \underbrace{b \cdot (\underline{c}\, e_{123})}_{(\vec{b} \;\; \underline{c})\mathbf{e}_{123}} = \beta \; (-b \wedge \underline{c})\, e_{123} \overset{(88)}{=} \alpha\beta\beta \; e_{123}^2. \tag{93}$$

We further contract each side of (87) from the left with $\underline{c}$ to obtain

$$\begin{aligned}
\alpha\underline{c}^2 + \varepsilon_4\alpha \; & \underbrace{\underline{c} \cdot \underline{c}}_{-\alpha\alpha'} \\
&= \underline{c} \cdot [(\varepsilon_4\beta \; b \; - \beta b)\, e_{123}] \\
&= \varepsilon_4\beta \; (\underline{c} \wedge b \;)\, e_{123} - \beta(\underline{c} \wedge b)\, e_{123} \\
&\overset{(88),(89)}{=} -\varepsilon_4\alpha\beta \; ^2 \, e_{123}^2 + \alpha\beta^2 \, e_{123}^2,
\end{aligned} \tag{94}$$

or, equivalently,

$$\alpha\underline{c}^2 = \alpha[\varepsilon_4\alpha \; ^2 - \varepsilon_4\beta \; ^2 \, e_{123}^2 + \beta^2 \, e_{123}^2]. \tag{95}$$

For $\alpha \neq 0$, we similarly contract each side of (87) from the left with $\underline{c}$ to obtain

$$\begin{aligned}
\alpha \underbrace{\underline{c}\cdot\underline{c}}_{-\alpha\alpha'} + \varepsilon_4\alpha\,\underline{c}^{\,2}
&= \underline{c}\cdot[(\varepsilon_4\beta\, b - \beta b)\,\mathrm{e}_{123}] \\
&= \varepsilon_4\beta\,(\underline{c}\wedge b)\,\mathrm{e}_{123} - \beta(\underline{c}\wedge b)\,\mathrm{e}_{123} \\
&\overset{(90),(91)}{=} \varepsilon_4\alpha\,\beta^{2}\,\mathrm{e}_{123}^2 - \alpha\,\beta^2\,\mathrm{e}_{123}^2,
\end{aligned} \tag{96}$$

or equivalently ($\varepsilon_4^2 = 1$)

$$\alpha\,\underline{c}^{\,2} = \alpha\,[\varepsilon_4\alpha^2 + \beta^{2}\,\mathrm{e}_{123}^2 - \varepsilon_4\beta^2\,\mathrm{e}_{123}^2]. \tag{97}$$

The inner product of (84) with b leads to

$$\begin{aligned}
\alpha b^2 &= -\varepsilon_4 \underbrace{b\cdot(b\cdot\underline{c})}_{(\vec{b}\ \vec{b}')\cdot\underline{c}'} - \beta \underbrace{b\cdot(\underline{c}\,\mathrm{e}_{123})}_{(\vec{b}\ \underline{c})\,\mathbf{e}_{123}} \\
&= \varepsilon_4(\alpha\underline{c} + \alpha\,\underline{c})\cdot\underline{c} + \alpha\beta^2\,\mathrm{e}_{123}^2 \\
&= \varepsilon_4\alpha\underline{c}^{\,2} + \varepsilon_4\alpha \underbrace{\underline{c}\cdot\underline{c}}_{-\alpha\alpha'} + \alpha\beta^2\,\mathrm{e}_{123}^2 \\
&= \alpha[\varepsilon_4\underline{c}^{\,2} - \varepsilon_4\alpha^{2} + \beta^2\,\mathrm{e}_{123}^2]
\end{aligned} \tag{98}$$

where we inserted (86) and (88) for the second equality. Assuming $\alpha \neq 0$, equation (98) leads with (97) to

$$\begin{aligned}
\alpha b^2 &= \varepsilon_4\alpha[\varepsilon_4\alpha^2 + \beta^{2}\,\mathrm{e}_{123}^2 - \varepsilon_4\beta^2\,\mathrm{e}_{123}^2] - \varepsilon_4\alpha\alpha^{2} + \alpha\beta^2\,\mathrm{e}_{123}^2 \\
&= \alpha[\alpha^2 - \varepsilon_4\alpha^{2} + \varepsilon_4\beta^{2}\,\mathrm{e}_{123}^2],
\end{aligned} \tag{99}$$

The inner product of (85) with b leads to

$$\begin{aligned}
\alpha b^{2} &= -\underbrace{b\cdot(b\cdot\underline{c})}_{(\vec{b}'\ \vec{b})\cdot\underline{c}'} - \beta\underbrace{b\cdot(\underline{c}\,\mathrm{e}_{123})}_{(\vec{b}'\ \underline{c})\,\mathbf{e}_{123}} \\
&= -(\alpha\underline{c} + \alpha\,\underline{c})\cdot\underline{c} + \alpha\beta^{2}\,\mathrm{e}_{123}^2 \\
&= -\alpha\underline{c}
\end{aligned}$$

Inserting (95), (98), and (100) into the root equation (73) for $\alpha \neq 0$ we obtain (for all α)

$$\begin{aligned}
&\alpha^2 + b^2 + \underline{c}^2 + \beta^2\, \mathrm{e}_{123}^2 + \varepsilon_4\alpha^{\ 2} - \varepsilon_4 b^{\ 2} + \varepsilon_4\underline{c}^{\ 2} - \varepsilon_4\beta^{\ 2}\, \mathrm{e}_{123}^2 \\
&= \alpha^2 + \varepsilon_4\underline{c}^{\ 2} - \varepsilon_4\alpha^{\ 2} + \beta^2\, \mathrm{e}_{123}^2 + \varepsilon_4\alpha^{\ 2} - \varepsilon_4\beta^{\ 2}\, \mathrm{e}_{123}^2 + \beta^2\, \mathrm{e}_{123}^2 + \beta^2\, \mathrm{e}_{123}^2 \\
&+ \varepsilon_4\alpha^{\ 2} + \varepsilon_4\underline{c}^{\ 2} - \varepsilon_4\alpha^{\ 2} - \varepsilon_4\beta^{\ 2}\, \mathrm{e}_{123}^2 + \varepsilon_4\underline{c}^{\ 2} - \varepsilon_4\beta^{\ 2}\, \mathrm{e}_{123}^2 \\
&= \alpha^2 + 3\alpha^2 - 3\alpha^2 + 3\varepsilon_4\underline{c}^{\ 2} + 3\beta^2\, \mathrm{e}_{123}^2 - 3\varepsilon_4\beta^{\ 2}\, \mathrm{e}_{123}^2 \\
&= 4\alpha^2 + 3\varepsilon_4[\underline{c}^{\ 2} - \varepsilon_4\alpha^2 - \beta^{\ 2}\, \mathrm{e}_{123}^2 + \varepsilon_4\beta^2\, \mathrm{e}_{123}^2] = -1, \qquad (102)
\end{aligned}$$

If in addition $\alpha \ \neq 0$ then with (97) we get for the root equation

$$\begin{aligned}
&\alpha^2 + b^2 + \underline{c}^2 + \beta^2\, \mathrm{e}_{123}^2 + \varepsilon_4\alpha^{\ 2} - \varepsilon_4 b^{\ 2} + \varepsilon_4\underline{c}^{\ 2} - \varepsilon_4\beta^{\ 2}\, \mathrm{e}_{123}^2 \\
&= 4\alpha^2 + 0 = -1, \qquad (103)
\end{aligned}$$

Therefore, we have no solution for $\alpha \neq 0$ and $\alpha \ \neq 0$.

7.1. $n = 4$, $\alpha \neq 0$, $\alpha \ = 0$

In this case constraints (83) – (89) become

$$\underline{c} \cdot \underline{c} \ = 0, \qquad (104)$$

$$\alpha b = -\varepsilon_4 b \ \cdot \underline{c} \ - \beta\underline{c}\, \mathrm{e}_{123}, \qquad (105)$$

$$\alpha b \ = -b \cdot \underline{c} \ - \beta \ \underline{c}\, \mathrm{e}_{123}, \qquad (106)$$

$$\underline{c} \ = \frac{1}{\alpha} b \ \wedge b, \qquad (107)$$

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We calculate from (108) that

$$\begin{aligned}
\alpha^2 \underline{c}^2 &= (\varepsilon_4 \beta\, b\;\; - \beta b)^2\, e_{123}^2 \\
&= (\beta^{\,2} b^{\,2} + \beta^2 b^2 - 2\varepsilon_4 \beta\beta\, b\;\; \cdot b)\, e_{123}^2 \\
&\overset{(111),(113),(114)}{=} [\beta^{\,2}(-\underline{c}^{\,2} + \beta^{\,2}\, e_{123}^2) + \beta^2(\varepsilon_4 \underline{c}^{\,2} + \beta^2\, e_{123}^2) - 2\varepsilon_4 \beta\beta\, (\beta\beta\;\; e_{123}^2)]\, e_{123}^2 \\
&= \underline{c}^{\,2}(-\beta^{\,2} + \varepsilon_4 \beta^2)\, e_{123}^2 + \beta^{\,4} + \beta^4 - 2\varepsilon_4 \beta^2 \beta^{\,2} \\
&= \underline{c}^{\,2}(-\beta^{\,2} + \varepsilon_4 \beta^2)\, e_{123}^2 + [(-\beta^{\,2} + \varepsilon_4 \beta^2)\, e_{123}^2]^2. \qquad (115)
\end{aligned}$$

Inserting (112) in (115) we get

$$\alpha^2 \underline{c}^2 = \varepsilon_4 \underline{c}^2 \underline{c}^{\,2} + \underline{c}^4. \qquad (116)$$

If $\underline{c}^2 \neq 0$ in (116) then

$$\varepsilon_4 \alpha^2 = \underline{c}^{\,2} + \varepsilon_4 \underline{c}^2, \qquad (117)$$

and the root equation (102) becomes with (112)

$$4\alpha^2 + 3\varepsilon_4[\underline{c}^{\,2} - \varepsilon_4\alpha^2 - \beta^{\,2}\, e_{123}^2 + \varepsilon_4 \beta^2\, e_{123}^2] = 4\alpha^2 = -1, \qquad (118)$$

= 0 in c

which has no solution for real 2 $\varepsilon_4 = 1$, = 2 b^0 −

7.2. $n = 4$, $\alpha = 0$, $\alpha \ \neq 0$

For $\alpha = 0$ the root equation (73) simplifies to

$$b^2 + \underline{c}^2 + \beta^2\, \mathrm{e}_{123}^2 + \varepsilon_4\alpha\ {}^2 - \varepsilon_4 b\ {}^2 + \varepsilon_4\underline{c}\ {}^2 - \varepsilon_4\beta\ {}^2\, \mathrm{e}_{123}^2 = -1, \tag{124}$$

The constraint equations (83) – (89) which have to be satisfied become

$$\underline{c} \cdot \underline{c}\ = 0, \tag{125}$$

$$b\ \cdot \underline{c}\ = -\varepsilon_4\beta\underline{c}\ \mathrm{e}_{123}, \tag{126}$$

$$b \cdot \underline{c}\ = -\beta\ \underline{c}\ \mathrm{e}_{123}, \tag{127}$$

$$b \wedge \underline{c} = 0, \tag{128}$$

$$b\ \wedge \underline{c} = 0. \tag{129}$$

$$\alpha\ \underline{c} = b\ \wedge b, \tag{130}$$

$$\alpha\ \underline{c}\ = (\beta\ b\ - \varepsilon_4\beta b)\, \mathrm{e}_{123}. \tag{131}$$

Especially for $\alpha\ \neq 0$ we obtain from (130) and (131) the constraints

$$\underline{c} = \frac{1}{\alpha} b\ \wedge b, \tag{132}$$

$$\underline{c}\ = \frac{1}{\alpha}(\beta\ b\ - \varepsilon_4\beta b)\, \mathrm{e}_{123}. \tag{133}$$

It is obvious that with (132) equations (128) and (129) are then fulfilled, because

$$b \wedge b\ \wedge b = 0 \text{ and } b\ \wedge b\ \wedge b = 0. \tag{134}$$

Due to (134) equation (125) is also fulfilled

$$\begin{aligned} \underline{c} \cdot \underline{c} &\overset{(132)}{=} \frac{1}{\alpha\ {}^2}(b\ \wedge b) \cdot [(\beta\ b\ - \varepsilon_4\beta b)\, \mathrm{e}_{123}] \\ &= \frac{1}{\alpha\ {}^2}[\beta\ (b\ \wedge b \wedge b\ )\, \mathrm{e}_{123} - \varepsilon_4\beta(b\ \wedge b \wedge b)\, \mathrm{e}_{123}] = 0. \end{aligned} \tag{135}$$

Using (133) we now check the remaining (126) and (127)

$$\begin{aligned} b\ \cdot \underline{c}\ &= \frac{1}{\alpha} b\ \cdot [(\beta\ b\ - \varepsilon_4\beta b)\, \mathrm{e}_{123}] \\ &= \frac{1}{\alpha}[\beta\ \underbrace{b\ \wedge b\ }_{=0}\ \mathrm{e}_{123} - \varepsilon_4\beta \underbrace{b\ \wedge b}_{=\alpha'\underline{c}}\ \mathrm{e}_{123}] \overset{(130)}{=} -\varepsilon_4\beta\underline{c}\ \mathrm{e}_{123}, \end{aligned} \tag{136}$$

$$\begin{aligned} b \cdot \underline{c}\ &= \frac{1}{\alpha} b \cdot [(\beta\ b\ - \varepsilon_4\beta b)\, \mathrm{e}_{123}] \\ &= \frac{1}{\alpha}[\beta\ \underbrace{b \wedge b\ }_{=-\alpha'\underline{c}}\ \mathrm{e}_{123} - \varepsilon_4\beta \underbrace{b \wedge b}_{=0}\ \mathrm{e}_{123}] \overset{(130)}{=} -\beta\ \underline{c}\ \mathrm{e}_{123}. \end{aligned} \tag{137}$$

Therefore, if the two *constraints* (132) and (133) are satisfied, all other necessary equations are also satisfied and the root equation depends only on α , β, β , b, and b :

$$b^2 + \frac{1}{\alpha^{2}}(b \wedge b)^2 + \beta^2 \mathrm{e}_{123}^2 + \varepsilon_4 \alpha^{2}$$
$$- \varepsilon_4 b^{2} + \varepsilon_4 \frac{1}{\alpha^{2}}(\beta b - \varepsilon_4 \beta b)^2 \mathrm{e}_{123}^2 - \varepsilon_4 \beta^{2} \mathrm{e}_{123}^2 = -1. \qquad (138)$$

7.3. $n = 4$, $\alpha = \alpha = 0$

For $\alpha = \alpha = 0$ the root equation (73) simplifies to

$$b^2 + \underline{c}^2 + \beta^2 \mathrm{e}_{123}^2 - \varepsilon_4 b^{2} + \varepsilon_4 \underline{c}^{2} - \varepsilon_4 \beta^{2} \mathrm{e}_{123}^2 = -1, \qquad (139)$$

The constraint equations (83) – (89) which have to be satisfied become

$$\underline{c} \cdot \underline{c} = 0, \qquad (140)$$

$$b \cdot \underline{c} = -\varepsilon_4 \beta \underline{c}\, \mathrm{e}_{123}, \qquad (141)$$

$$b \cdot \underline{c} = -\beta\, \underline{c}\, \mathrm{e}_{123}, \qquad (142)$$

$$b \wedge \underline{c} = 0, \qquad (143)$$

$$b \wedge \underline{c} = 0. \qquad (144)$$

$$b \wedge b = 0, \qquad (145)$$

$$\beta b = \varepsilon_4 \beta b. \qquad (146)$$

Case: $\beta = 0, \beta \neq 0$
Now only the constraints

$$\underline{c} \cdot \underline{c} = 0, \qquad b \cdot \underline{c} = -\beta\, \underline{c}\, \mathrm{e}_{123}, \qquad b \wedge \underline{c} = 0 \tag{155}$$

remain. The second identity in (155) is equivalent to the *constraint*

$$\underline{c} = -\frac{1}{\beta}\, b \cdot \underline{c}\ \mathrm{e}_{123}^{-1}. \tag{156}$$

We can check that based on (156) the other two constraints of (155) are also satisfied

$$\underline{c} \cdot \underline{c} = -\frac{1}{\beta}\,[b \cdot \underline{c}\ \mathrm{e}_{123}^{-1}] \cdot \underline{c} = -\frac{1}{\beta}\,[\underbrace{(b \cdot \underline{c}) \wedge \underline{c}}_{=0}]\ \mathrm{e}_{123}^{-1} = 0, \tag{157}$$

and

$$b \wedge \underline{c} = -\frac{1}{\beta}\, b \wedge [b \cdot \underline{c}\ \mathrm{e}_{123}^{-1}] = -\frac{1}{\beta}\,[(b \wedge b) \cdot \underline{c}\,]\ \mathrm{e}_{123}^{-1} = 0. \tag{158}$$

Inserting $\beta = 0$ and (156) into (147) yields the root equation

$$b^2 + \frac{1}{\beta^{2}}\,(b \cdot \underline{c}\,)^2\, \mathrm{e}_{123}^2 + \varepsilon_4 \underline{c}^{\,2} - \varepsilon_4 \beta^{\,2}\, \mathrm{e}_{123}^2 = -1. \tag{159}$$

Case: $\beta \neq 0$
Because of $\beta \neq 0$, the constraints (148) – (152) reduce to

$$b = 0, \qquad \underline{c} = 0. \tag{160}$$

and the root equation becomes

$$\beta^2\, \mathrm{e}_{123}^2 + \varepsilon_4 \underline{c}^{\,2} - \varepsilon_4 \beta^{\,2}\, \mathrm{e}_{123}^2 = -1. \tag{161}$$

7.3.2. $n = 4$, $\alpha = \alpha = 0$, $b \neq 0$.

Case: $b = 0, \beta = 0$
This reduces equations (140) – (146) to the constraints

$$\beta = 0, \qquad \underline{c} \cdot \underline{c} = 0, \qquad b \cdot \underline{c} = 0, \qquad b \wedge \underline{c} = 0, \tag{162}$$

The root equation (139) becomes then

$$\underline{c}^2 - \varepsilon_4 b^{\,2} + \varepsilon_4 \underline{c}^{\,2} = -1. \tag{163}$$

Case: $b = 0, \beta \neq 0$
This reduces the constraint equations (140) – (146) to

$$\underline{c} \cdot \underline{c} = 0, \tag{164}$$

$$b \cdot \underline{c} = -\varepsilon_4 \beta \underline{c}\, \mathrm{e}_{123} \implies \underline{c} = -\frac{\varepsilon_4}{\beta}\, b \cdot \underline{c}\ \mathrm{e}_{123}^{-1}, \tag{165}$$

$$\beta\, \underline{c} = 0, \tag{166}$$

$$b \wedge \underline{c} = 0. \tag{167}$$

$$\beta b = 0 \implies \beta = 0. \tag{168}$$

We now check the remaining four constraints (140), (141), (143), (144) for consistency. Due to the proportionality (181) of b and b , (143) and (144) are seen to be equivalent. Inserting (182) into the right hand side of (141) gives

$$-\varepsilon_4 \beta \frac{(-1)}{\beta} b \cdot \underline{c} \;\; \mathrm{e}_{123}^{-1} \; \mathrm{e}_{123} = \varepsilon_4 \frac{\beta}{\beta} b \cdot \underline{c} \overset{(181)}{=} b \;\; \cdot \underline{c}$$

Table 1. Geometric roots of -1 for Clifford algebras $C\ell_{p,q}$, $n = p+q \leq 3$. The multivectors are denoted for $n=1$ by $\alpha + \beta e_1$, for $n=2$ by $\alpha + b_1 e_1 + b_2 e_2 + \beta \mathbf{e}_{12}$, and for $n=3$ by $\alpha + b_1 e_1 + b_2 e_2 + b_3 e_3 + c_1 \mathbf{e}_{23} + c_2 \mathbf{e}_{31} + c_3 \mathbf{e}_{12} + \beta \mathbf{e}_{123}$.

n	Cases	Solutions A and root equations
1		no solution for $C\ell_{1}$ $A = \pm e_1$ for $C\ell_{0,1}$
2	$\alpha = 0$	$\beta^2 = b_1^2\varepsilon_2 + b_2^2\varepsilon_1 + \varepsilon_1\varepsilon_2$ $\beta^2 = \begin{cases} b_1^2 + b_2^2 + 1 & \text{for } C\ell_{2} \\ -b_1^2 + b_2^2 - 1 & \text{for } C\ell_{1,1} \\ -b_1^2 - b_2^2 + 1 & \text{for } C\ell_{0,2} \end{cases}$
	$\alpha \neq 0$	no solution
3	Constraint: $\alpha = \beta = 0$	$0 = b \wedge \underline{c} = b_1c_1 + b_2c_2 + b_3c_3$ $-1 = b^2 + \underline{c}^2$ $-1 = b_1^2\varepsilon_1 + b_2^2\varepsilon_2 + b_3^2\varepsilon_3 - c_1^2\varepsilon_2\varepsilon_3 - c_2^2\varepsilon_3\varepsilon_1 - c_3^2\varepsilon_1\varepsilon_2$ $-1 = \begin{cases} b_1^2 + b_2^2 + b_3^2 - (c_1^2 + c_2^2 + c_3^2) & \text{for } C\ell_{3} \\ b_1^2 - b_2^2 - b_3^2 - (c_1^2 - c_2^2 - c_3^2) & \text{for } C\ell_{1,2} \\ b_1^2 + b_2^2 - b_3^2 + (c_1^2 + c_2^2 - c_3^2) & \text{for } C\ell_{2,1} \\ -(b_1^2 + b_2^2 + b_3^2) - (c_1^2 + c_2^2 + c_3^2) & \text{for } C\ell_{0,3} \end{cases}$
	$\alpha = 0, \beta \neq 0$	$A = \pm \mathbf{e}_{123}$ for $C\ell_{3}$, $C\ell_{1,2}$ no solution for $C\ell_{2,1}$, $C\ell_{0,3}$
	$\alpha \neq 0$	no solution

- How can the graded structure of $C\ell_{p,q}$ be used best in the calculation of higher order geometric multivector square roots of -1? This also includes a question how to best use, for this type of computation, invariance of the equation $AA = -1$ under Clifford algebra (anti) automorphisms such as grade involution, reversion or conjugation, and under symmetries of the root equation. For example, under the grade involution,

$$AA = -1 \iff \hat{A}\hat{A} = -1. \tag{187}$$

Table 2. Geometric roots of -1 for Clifford algebras $Cl_{p,q}$, $n = p + q = 4$. The multivectors are denoted by $\alpha + b + c$

periodicity of Cli□ord algebras and the isomorphisms with matrix rings. Central elements squaring to -1 would be of particular importance as then they can be used in place of the imaginary *i*.

- The further use of Cli□ord algebra computation software like CLIFFORD for MAPLE and other packages [22, 24, 25].

Of special interest in physics are the Cli□ord algebras of Minkowski space-time, sometimes called [17] *space-time algebras* $C_{3,1}$ and $C_{1,3}$. Table 2 contains the complete set of all geometric roots of -1 for these algebras, so in particular all possible geometric multivector elements that may take on the role of the imaginary unit *i* in quantum mechanics, which is e.g. fundamental for the description of spin and for wave propagation.

Finally, the door is now wide open to construct all possible new types of Cli□ord Fourier transformations (CFT) [26] for multivector fields with domains and image domains ranging over the full Cli□ord algebras involved or subalgebras and subspaces thereof. In particular all known Fourier transformations will find their place in this new general framework. The close relationship of wavelet transformations [27] and windowed transformations [28] to Fourier transformations shows that also in these fields new mathematics is to be expected.

Examples of CFTs working with non-central replacements of the imaginary unit *i* [illegible] quaternion FT (QFT) [5, 13, 14, 29], and the CFT [9, 12] where

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