DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

SCHUR POLYNOMIALS AND THE IRREDUCIBLE REPRESENTATIONS OF $\mathcal{S}_n$

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1 Introduction

Schur polynomials are certain homogeneous symmetric polynomials in n indeterminates with integer coefficients and correspond to the irreducible representations of S_n. One of the main problems in the field of representation theory is the decomposition of a representation into irreducible components realized as irreducible modules.

The Littlewood-Richardson rule can be used to find the irreducible modules of the symmetric group, or in a general linear group it can be used to find the decomposition of a tensor product into irreducibles, in both cases by looking at the corresponding Schur functions. An overview of the representation of the symmetric group will be presented, but the focus of this paper is on the calculation of Littlewood-Richardson coefficients using the Littlewood-Richardson rule.

Throughout this paper we denote a partition λ of positive integer n by λ n, where λ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and the λ_i are weakly decreasing positive nonzero integers and $\sum_{i=i}^{l}$)*and the*

Definition 1. *The symmetric group, denoted S_n, is the group consisting of all bijections from $\{1, 2, \ldots, n\}$ to itself under the operation of composition. The elements of S_n are permutations, which we multiply from right to left. That is, $\pi\sigma$ means to apply permutation σ first and then apply permutation π.*

The notation we will use for the permutations of S_n is **cycle notation**. For example, take the permutation $\pi = (142)(36)(5)$. This notation tells us that the permutation π maps 1 to 4, 4 to 2, 2 to 1, 3 to 6, and 5 to itself. Notice that π in our example consists of 3 disjoint cycles, and since disjoint cycles commute, reordering the cycles does not change the permutation. That is, $(142)(36)(5) = (36)(142)(5) = (5)(36)(142)$.

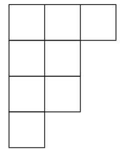
Definition 2. *A **k-cycle** or **cycle of length k**, is a cycle containing k elements.*

For example, $\pi = (142)(36)(5)$ contains a 3-cycle, 2-cycle, and a 1-cycle. We will find that cycles of particular lengths, or cycle types, determine conjugacy classes of the symmetric group S_n. Since these conjugacy classes are intimately connected with Schur polynomials, we will introduce Schur polynomials before discussing the symmetric group further. The definitions, theorems, and propositions in this paper are taken from [2] unless otherwise indicated. Some proofs have been omitted, but they can be found in [2] as well.

2 Schur Polynomials

Schur polynomials, also called Schur functions, arise in many different contexts. These polynomials form a basis for the space of all symmetric polynomials. There are different ways to define Schur polynomials. A Schur polynomial depends on a partition λ of a positive integer d. One definition of these polynomials as they arise from antisymmetric polynomials will be presented later in this paper. For now we introduce Schur polynomials using a combinatorial approach with Young tableaux, which also play an important role in the Littlewood-Richardson rule. We will see that Young tableaux depend on a given partition λ as well, and hence there is a natural relationship bee Schur [illegible] rmr exa[illegible] : [5m w27(u173[illegible]0n[x[2])-326(as4.3463(partit3

Example 1. The partition $\lambda = (3, 2, 2, 1)$ has Ferrers diagram



Definition 4. *A* ***filling*** *of a Ferrers diagram is any way of putting a positive integer in each box of the diagram. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a filling of a Ferrers diagram. Each μ_i is the number of times the integer i appears in the diagram.*

Example 2. A possible tableau for $\lambda = (3, 2, 2, 1)$ with filling $\mu = (2, 4, 2)$ is

1	2	3
2	2	
3	2	
1		

Notice that in order for the diagram to be completely filled, it is necessary for $|\lambda| = |\mu|$.

While it is possible to fill tableaux arbitrarily in this manner, the tableaux will be more useful if we impose restrictions on the filling μ. These restrictions lead us to the definition of Young tableaux.

Definition 5. *Suppose λ n. A* ***Young tableau of shape*** *λ, is an array T obtained by filling the Ferrers diagram of shape λ so that the filling is weakly increasing across each row and strictly increasing down each column. A Young tableau of shape λ is also called a λ-tableau. A tableau T is* ***standard*** *if the rows and columns of T are increasing sequences. That is, T is filled with the numbers $1, 2, \ldots, n$ bijectively. A tableau T is* ***semistandard*** *if the filling is weakly increasing across each row and strictly increasing down each column.*

Example 3. Given $\lambda = (3, 2, 2, 1)$ as above, a standard tableau T would be

1	2	3
4	5	
6	7	
8		

Suppose we relax the necessary conditions of a standard tableau by allowing repetition, which leads to a semistandard tableau. Then a possible semistandard tableau is

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Notice that 1 and 4 appear twice, and 2 appears three times. The filling in this example is $\mu = (2, 3, 1, 2)$.

Recall the definition of a filling $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. In order for a tableau to be semistandard, the number of boxes in the first column must be less than k. For example, given $\mu = (2, 3)$, it is not possible to have a corresponding semistandard tableau of shape $\lambda = (3, 1, 1)$. Although $|\mu| = |\lambda|$, it is not possible to fill the three boxes of the first column in a strictly increasing manner with the given μ.

This paper will focus on semistandard tableaux since this type of tableaux gives rise to Schur polynomials and also appears in the Littlewood-Richardson rule.

Now we are ready to define the Schur polynomial determined by a partition λ and the corresponding Young tableaux.

Definition 6. *Fix λ and a bound N on the size of the entries in each semistandard tableau T. Let $x^T = \prod_{i=1}^{N} x_i^j$, where $j_{\square}$= the number of i's in T. Then the* **Schur polynomial** *is* $s_\lambda(x_1, \ldots x_N) := \sum_{\text{semistandard } T} x^T$.

Example 4. Let $\lambda = (2, 1)$. Then the list of possible semistandard tableaux of shape λ where $N = 3$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

The corresponding Schur polynomial is given by

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3 x_2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

The Schur polynomial $s_{(2,1)}$ is a homogeneous symmetric polynomial of degree $|\lambda| = 3$. Notice that each term corresponds to one possible tableau, and the degree of each term is three which corresponds to the number of boxes in the given tableau, i.e., $|\lambda|$. To understand homogeneous symmetric polynomials, we now introduce symmetric functions.

2.2 Introduction to Symmetric Functions

Schur functions, as special symmetric polynomials, have been introduced. There are other types of symmetric functions as well, among them elementary symmetric functions and power sum symmetric functions. Each of these families of symmetric functions forms a basis for the vector space of symmetric functions.

Definition 7. *A polynomial* $f \quad \mathbb{C}[x_1, x_2, \ldots, x_n]$ *is a* ***symmetric polynomial*** *if it is invariant under any permutation* σ *of the subscripts* $1, 2, \ldots, n$*. That is,*

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n)$$

for any $\sigma \quad S_n$*.*

Symmetric polynomials are a subring of the ring of multivariate polynomials. Let the ring of symmetric polynomials be denoted $\mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}$.

We also say that a monomial x^λ

Every symmetric polynomial can be written uniquely in terms of elementary symmetric polynomials. Notice that each σ_r

In words, any antisymmetric polynomial is invariant under all even permutations and not fixed by any odd permutations of S_n. Moreover, these polynomials change sign when acted upon by an odd permutation.

Now we show that any antisymmetric polynomial is divisible by D, where

$$D = \prod_{1 \le i < j \le n} (x_i - x_j),$$

and the result is a symmetric polynomial.

Proposition 1. *Every polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]^{A_n}$ can be written uniquely in the form $f = g + hD$ where g and h are symmetric polynomials.*

Proof. Let

$$g(x_1, x_2, \ldots, x_n) := \frac{1}{2}[f(x_1, x_2, \ldots, x_n] + f(x_2, x_1, \ldots, x_n]$$

and

$$\overline{h}(x_1, x_2, \ldots, x_n) := \frac{1}{2}[f(x_1, x_2, \ldots, x_n) - f(x_2, x_1, \ldots, x_n)],$$

where g is a symmetric polynomial and $\overline{h}$ is an antisymmetric polynomial for all $\sigma \in S_n$. That is,

$$\overline{h}(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = \text{sign}(\sigma) \cdot \overline{h}(x_1, x_2, \ldots, x_n).$$

Since $\overline{h}$ is antisymmetric, $\overline{h}$ vanishes when one variable, say x_i, is replaced with a different variable x_j. Then $x_i - x_j$ divides $\overline{h}$ for all $1 \le i < j \le n$. Since $x_i - x_j$ divides, $\overline{h}$, D divides $\overline{h}$.

h D To show uniqueness, divides

$$a_\lambda(x_1, \ldots, x_n) = \det \begin{pmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \ldots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_l} & x_2^{\lambda_l} & \ldots & x_n^{\lambda_n} \end{pmatrix}$$

The degree of the polynomial $a_\lambda = d + \binom{n}{2}$. The polynomials a_λ form a basis for the vector space of all antisymmetric polynomials, and hence we can divide

```
> s2 := factor(a2)/Dee;
```

$$s2 := x_3\,x_2 + x_1\,x_2 + x_1\,x_3$$

```
> c2:=-1:  f2 := f1-c2*s2;
```

$$f2 := 0$$

```
> evalb(f = c1*s1+c2*s2);
```

$$true$$

Note that f written in terms of Schur polynomials is $f = s1 - s2$ where $s1$ and $s2$ are given below.

```
> s1; s2;
```

$$x_2^2 + x_3\,x_2 + x_1\,x_2 + x_3^2 + x_1\,x_3 + x_1^2$$
$$x_3\,x_2 + x_1\,x_2 + x_1\,x_3$$

3 S_n and Conjugacy Classes

Recall the symmetric group S_n and cycle notation. It turns out that the cycle notation used to describe permutations of S_n can be referenced by cycle type and also be in correspondence with a partition λ of the positive integer n. This correspondence is how Schur polynomials can be related to the representations of S_n.

Definition 12. *If π S_n is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$, then the integers $n_1, n_2, \ldots, n_r$ are the* **cycle type** *of π.*

Definition 13. *In any group G, the elements g and h are* **conjugates** *if*

$$g = khk^{-1}$$

for some k G. The set of all elements conjugate to a given g is called the **conjugacy class of g**. *Conjugacy classes partition the group G.*

Let λ correspond to the cycle type of a permutation σ S_n. Then we see a one-to-one correspondence between partitions of n and the conjugacy classes of S_n.

Example 8. For example, consider S_4. Below are the partitions of 4 and the corresponding conjugacy classes of S_4.

$$\lambda_1 = (1, 1, 1, 1) \rightarrow \{$$

Notice there are five partitions of 4 that correspond to the five conjugacy classes of S_4.

Since each Schur polynomial s_λ corresponds to the partition λ, the s_λ corresponds to the conjugacy class of S_n described by λ.

4 Representations

In this section we define representations and describe what it means for a representation to be irreducible. There are different ways to represent a group, but here we will introduce a representation of a group as a group of matrices. Matrix representations are easily manipulated, and lend themselves well to the concept of modules which will be presented in this section. Representations of a group G are commonly discussed in terms of G-modules.

While matrices are easy to manipulate, a group of matrices can be cumbersome to deal with. In this section we introduce characters which are the traces of the matrices associated with a representation. Characters are useful in analyzing representations. Also in this section we introduce a special relationship between representations of S_n and the representations of its subgroups which will be important later in our discussion of the Littlewood-Richardson rule.

4.1 Definitions

In this section we introduce definitions and propositions regarding representations and irreducibility in order to understand what it means to decompose a representation into its irreducible components.

Definition 14. *Let Mat_d be the set of all $d \times d$ matrices with entries in $\mathbb{C}$. Let the* **complex general linear group of degree d**, *denoted GL_d, be the group of all invertible matrices $X = (x_{i,j})_{d\times d}$. Then a* **matrix representation of a group G** *is a group homomorphism*

$$X : G \to GL_d.$$

Equivalently, to each $g \in G$ is assigned a matrix $X(g)$ such that

1. *$X(e) = I$ (the identity matrix), and*
2. *$X(gh) = X(g)X(h)$ for all $g, h \in G$.*

The parameter d is called the **degree** *or* **dimension** *of the representation.*

By mapping every element of a group G to the identity matrix, we find that every group has a trivial representation. Recall Cayley's theorem which states that every finite group is isomorphic to a permutation group. It can be shown that every finite group has a permutation matrix representation. Since we are focusing on the symmetric group, we will assume that all groups mentioned from here on are finite and hence have a matrix representation. Let us define what it means for a representation to be irreducible. To do this we describe the concept of a module.

Definition 15. *Let V be a vector space and G be a group. Let general linear group of V, denoted $GL(V)$, be the set of all invertible linear transformations of V to itself. If dim $V = d$, then $GL(V)$ and GL_d are isomorphic as groups. We say V is a* ***G-module*** *if there is a group homomorphism*

$$\rho : G \to GL(V).$$

Definition 16. *Let V be a G-module. A* ***submodule of*** *V is a subspace W that is closed under the action of G, that is,*

$$w \in W \Rightarrow gw \in W \text{ for all } g \in G.$$

We also say that W is a G-invariant subspace, and W is a G-module in its own right.

Now we can define what it means for a representation to be irreducible.

Definition 17. *A non-zero G-module V is* ***reducible*** *if it contains a non-trivial submodule W. Otherwise V is said to be* ***irreducible***.

Theorem 1 (Maschke's Theorem). *Let G be a finite group and let V be a nonzero G-module. Then*

$V = W^{(1)}$ [illegible]

Proposition 2. *Let G, V, and W be as described in Maschke's Theorem above, where*

$$V = W^{(1)} \quad W^{(2)} \quad \cdots \quad W^{(k)}.$$

Then the number of pairwise inequivalent irreducible $W^{(i)}$ *equals the number of conjugacy classes of G.*

Thus the number of conjugacy classes of S_n gives the exact number of irreducible representations of S. In a previous Example 8, the five conjugacy classes of S_4 were presented, along with each corresponding λ. By Maschke's Theorem and corresponding propositions, S_4

(c) If X and Y are both representations of G, then $Y = TXT^{-1}$ for some fixed matrix T. Since the trace is invariant under conjugation, for all $g \in G$,

$$\operatorname{tr} Y(g) = \operatorname{tr} TX(g)T^{-1} = \operatorname{tr} X(g).$$

Definition 20. *Let χ and ψ be any two functions from a group G to the complex numbers $\mathbb{C}$. The* **inner product** *of characters χ and ψ is*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

This definition can be put into another form that is useful.

Proposition 4. *Let χ and ψ be characters. Then*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

The following proposition relates representations to characters, allowing us to compare representations by comparing their corresponding characters.

Proposition 5. *Let X be a matrix representation of G with character χ. Suppose*

$$X \cong m_1 X^{(1)} \oplus m_2 X^{(2)} \oplus \dots \oplus m_k X^{(k)},$$

where the $X^{(i)}$ are pairwise inequivalent irreducibles with characters $\chi^{(i)}$.

1. $\langle \chi, \chi \rangle = m_1^2 + m_2^2 + \dots + m_k^2$.
2. *X is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*
3. *Let Y be another matrix representation of G with character ψ. Then*

$$X \cong Y \text{ if and only if } \chi(g) = \psi(g) \text{ for all } g \in G.$$

These properties of characters will be used to prove an important theorem and proposition in describing representations. First we give some background on representation theory of the symmetric group.

4.3 Induced Representations of the Symmetric Group

where $\psi(g) = 0$ if $g \notin H$. Similarly,

$$\varphi(g) = \sum_i \psi(s_i^{-1} g s_i).$$

Since t_i and s_i are both traversals, we can permute subscripts if necessary and obtain $t_i H = s_i H$ for all i. Now $t_i = s_i h_i$, where $h_i \in H$ for all i, so $t_i^{-1} g t_i = h_i s_i^{-1} g s_i h_i$. This implies that $t_i^{-1} g t_i \in H$ if and only if $s_i^{-1} g s_i \in H$,

2. *If X, Y, and $X \quad Y$ have characters denoted by χ, ψ, and $\chi \quad \psi$, respectively, then*

$$(\chi \quad \psi)(g, h) = \chi(g)\psi(h$$

Using Proposition 4, we can simplify this equation to

$$\chi \quad \psi, \chi \quad \psi \; = \; \chi, \chi \;\; \psi, \psi \; .$$

By Proposition 5, we have that

$$\chi \quad \psi, \chi \quad \psi \; = 1 \cdot 1 = 1.$$

So we have that $\chi \quad \psi, \chi \quad \psi \; = 1$ and by Proposition 5, the character $\chi \quad \psi$ is irreducible. □

Proof. (2) Let $X^{(i)}$ and $Y^{(j)}$ have characters $\chi^{(i)}$ and $\psi^{(j)}$, respectively. Following the proof in part (1) we have

$$\chi^{(i)} \quad \psi^{(j)}, \chi^{(k)} \quad \psi^{(l)} \quad = \quad \chi^{(i)}, \chi^{(k)} \quad \psi^{(j)}, \psi^{(l)} \quad = \; \delta_{i,k}\delta_{j,l}.$$

Then by Proposition 5 we see that $\chi^{(i)} \quad \psi^{(j)}$ are pairwise inequivalent.

To prove that the list is complete, it is enough to show that the number of such representations is the number of conjugacy classes of $G \times H$. But the number of conjugacy classes of $G \times H$ is the number of conjugacy classes of G times the number of conjugacy classes of H, which is in turn the number of $X^{(i)} \quad Y^{(j)}$. □

Armed with knowledge of tableaux, Schur polynomials, and representations, we

Definition 24. *Let μ and v be arbitrary partitions. The* ***Littlewood-Richardson coefficients****, denoted $C^{\lambda}_{v\mu}$, are defined by*

$$s_v \cdot s_\mu = \sum_\lambda C^{\lambda}_{v,\mu} s_\lambda$$

where $|v| + |\mu| = |\lambda|$. These numbers count tensor product multiplicities of irreducible representations of $GL_d(\mathbb{C})$.

In the language of Specht modules, the Littlewood-Richardson coefficients are the $C^{\lambda}_{v,\mu}$, or multiplicities of Specht modules, in

$$(S^\mu \otimes S^v)\uparrow^{S_n} = \sum_\lambda C^{\lambda}_{v,\mu} S^\lambda$$

where $|\mu| + |v| = n$.

The Littlewood-Richardson coefficients $C^{\lambda}_{v,\mu}$ can be determined combinatorially by considering all possible fillings μ of the skew tableau λ/v subject to conditions described in the Littlewood-Richardson rule which will be introduced shortly. In order to define the Littlewood-Richardson rule we first explore skew tableaux and row lattice permutations.

Definition 25. *Let λ and v be Ferrers diagrams such that $v \subseteq \lambda$. The corresponding* ***skew diagram*** *is the set of cells*

$$\lambda/v = \{c : c \in \lambda \text{ and } c \notin$$

$\lambda \mu$

œ v=

Lattice permutations are also a method for encoding standard tableaux. Recall that standard tableaux have a bijective filling. That is, each element $1, 2, \ldots, n$ appears exactly once. Given a standard tableau T with n elements, form a sequence $\pi = i_1 i_2 \ldots i_n$ where $i_k = i$ if k appears in row i of T.

Example 11. Consider the tableau

1	4	5
2	6	7
3	8	

The corresponding lattice permutation is given by $\pi = 1\,2\,3\,1\,1\,2\,2\,3$. Notice that π corresponds to 1 in the first row, 2 in the second row, 3 in the third row, 4 in the first row, 5 in the first row, 6 and 7 in the second row, and 8 in the third row.

Similarly, given a lattice permutation we can construct the corresponding unique standard tableau. For example, the lattice permutation $\pi = 1\,2\,3\,1\,2\,3\,1\,2$ corresponds to

1	4	7
2	5	8
3	6	

Theorem 5 (Littlewood-Richardson Rule). *The value of the coe cient $C^{\lambda}_{v,\mu}$ is equal to the number of semistandard tableaux T such that*

(1) T has shape λ/v and content μ.

(2) A lattice permutation results from listing the entries of each skew tableau across each row from right to left and from top to bottom.

Let us look at an example.

Example 12. Let $v = (2, 0)$ and $\mu = (2, 1)$, and we will consider two variables x_1 and x_2. Recall that $|\lambda| = |v| + |\mu|$. In this example $|v| = |(2$

each r frmvλ|

To use the Littlewood-Richardson rule, we need to consider the skew tableaux corresponding to λ/v with all possible fillings μ where $v = (2, 0)$, $\mu = (2, 1)$, and λ can be $(5, 0)$, $(4, 1)$, or $(3, 2)$.

First let $\lambda = (5, 0)$. We are considering skew tableau $\lambda/(2, 0)$ with filling $\mu = (2, 1)$. The possible skew tableaux are

$A =$

Thus our problem is to find the coe□cients in

$$
\begin{aligned}
s_{(2,1,0)} \cdot s_{(2,1,0)} = {} & C^{(6,0,0)}_{(2,1,0),(2,1,0)} s_{(6,0,0)} + C^{(5,1,0)}_{(2,1,0),(2,1,0)} s_{(5,1,0)} \\
& + C^{(4,2,0)}_{(2,1,0),(2,1,0)} s_{(4,2,0)} + C^{(4,1,1)}_{(2,1,0),(2,1,0)} s_{(4,1,1)} + C^{(3,3,0)}_{(2,1,0),(2,1,0)} s_{(3,3,0)} \\
& + C^{(3,2,1)}_{(2,1,0),(2,1,0)} s_{(3,2,1)} + C^{(2,2,2)}_{(2,1,0),(2,1,0)} s_{(2,2,2)}
\end{aligned}
$$

When $\lambda = (6,0,0)$, $\nu = (2,1,0)$ is not contained within λ, so $C^{(6,0,0)}_{(2,1,0),(2,1,0)} = 0$. s

When $\lambda = (5,1,0)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 & 2 \\ \hline \bullet \\ \cline{1-1} \end{array}
\qquad
B = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & 1 & 2 & 1 \\ \hline \bullet \\ \cline{1-1} \end{array}
\qquad
C = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & 2 & 1 & 1 \\ \hline \bullet \\ \cline{1-1} \end{array}
$$

In this case, A is does not satisfy the lattice permutation condition, while B and C are not semistandard tableau. So $C^{(5,1,0)}_{(2,1,0),(2,1,0)} = 0$.

When $\lambda = (4,2,0)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 \\ \hline \bullet & 2 \\ \cline{1-2} \end{array}
\qquad
B = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 2 & 1 \\ \hline \bullet & 1 \\ \cline{1-2} \end{array}
\qquad
C = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 2 \\ \hline \bullet & 1 \\ \cline{1-2} \end{array}
$$

Then A is a semistandard tableau satisfying the lattice permutation condition with $\pi = 1\,1\,2$. B is not a semistandard tableau, and C does not satisfy the lattice permutation condition. So $C^{(4,2,0)}_{(2,1,0),(2,1,0)} = 1$.

When $\lambda = (4,1,1)$, the possible resulting skew tableaux are

$$
A = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 1 \\ \hline \bullet \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}
\qquad
B = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & 1 & 2 \\ \hline \bullet \\ \cline{1-1} \end{array}
$$

When $\lambda = (3, 2, 1)$, the possible resulting skew tableaux are

$$A = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 1 \\ \hline \bullet & 1 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \qquad B = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 1 \\ \hline \bullet & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} \qquad C = \begin{array}{|c|c|c|}\hline \bullet & \bullet & 2 \\ \hline \bullet & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Then B is not a semistandard tableaux. Both A and C are semistandard, but only A satisfies the lattice permutation condition with $\pi = 1\,1\,2$. So $C^{(3,2,1)}_{(2,1,0),(2,1,0)} = 2$.

When $\lambda = (2, 2, 2)$, the possible resulting skew tableaux are

$$A = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & 1 \\ \hline 1 & 2 \\ \hline\end{array} \qquad B = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & 2 \\ \hline 1 & 1 \\ \hline\end{array} \qquad C = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & 1 \\ \hline 2 & 1 \\ \hline\end{array}$$

Then A is a semistandard tableau satisfying the lattice permutation condition with $\pi = 1\,2\,1$. Both B and C are not semistandard tableaux. So $C^{(2,2,2)}_{(2,1,0),(2,1,0)} = 1$.

Putting all of this together, we find that

$$s_{(2,1,0)} \cdot s_{(2,1,0)} = s_{(4,2,0)} + s_{(4,1,1)} + s_{(3,3,0)} + 2s_{(3,2,1)} + s_{(2,2,2)}$$

In terms of Specht modules for $v = (2$

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7 Additional Reading

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