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Computation of Non-Commutative Gröbner Bases in Grassmann and Clifford Algebras

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Abstract. Tensor, Clifford and Grassmann algebras belong to a wide class of non-commutative algebras that have a Poincaré-Birkhoff-Witt (PBW) "monomial" basis. The necessary and sufficient condition for an algebra to have the PBW basis has been established by T. Mora and then V. Levandovskyy as the so called "non-degeneracy condition". This has led V. Levandovskyy to a re-discovery of the so called G-algebras (previously introduced by J. Apel) and GR-algebras (Gröbner-ready algebras). It was T. Mora who already in the 1990s considered a comprehensive and algorithmic approach to Gröbner bases for commutative and non-commutative algebras. It was T. Stokes who eighteen years ago introduced Gröbner left bases (GLB) and Gröbner left ideal bases (GLIB) for left ideals in Grassmann algebras, with the GLIB bases solving an ideal membership problem. Thus, a natural question is to first seek Gröbner bases with respect to a suitable admissible monomial order for ideals in tensor algebras T and then consider quotient algebras T/I. It was shown by Levandovskyy that these quotient algebras possess a PBW basis if and only if the ideal I has a Gröbner basis. Of course, these quotient algebras are of great interest because, in particular, Grassmann and Clifford algebras of a quadratic form arise this way. Examples of G-algebras include the quantum plane, universal enveloping algebras of finite dimensional Lie algebras, some Ore extensions, Weyl algebras and their quantizations, etc. Examples of GR-algebras, which are either G algebras or are isomorphic to quotient algebras of a G-algebra modulo a proper two-sided ideal, include Grassmann and Clifford algebras. After recalling basic concepts behind the theory of commutative Gröbner bases, a review of the Gröbner bases in PBW algebras, G-, and GR-algebras will be given with a special emphasis on computation of such bases in Grassmann and Clifford algebras. GLB and GLIB bases will also be computed.

Mathematics Subject Classification (2000). Primary 15A66, 15A75; Secondary 13P10, 16P40.

Keywords. PBW bases, monomial order, Gröbner GLB and GLIB bases, normal form, S-polynomial, Grassmann algebra, Clifford algebra, normal form.

2. Gröbner bases in polynomial rings

Our main reference is [19]. In particular, $k[x_1, \dots, x_n]$ is a polynomial ring in indeterminates $x_1, \dots, x_n$ over a field k whereas $\mathbf{V}(f_1, \dots, f_s)$ is an a□ne variety viewed as a subset of k^n consisting of common zeros of polynomials $f_1, \dots, f_s$ $k[x_1, \dots, x_n]$. In particular, $f_1, \dots, f_s$ denotes an ideal in $k[x_1, \dots, x_n]$ generated by the polynomials. We say that ideal I $k[x_1, \dots, x_n]$ is **finitely generated** if there exist $f_1, \dots, f_s$ $k[x_1, \dots, x_n]$ such that $I =$ $f_1, \dots, f_s$. Then we say that $f_1, \dots, f_s$ are a **basis** of I.

Proposition 2.1 (Cox). *If $f_1, \dots, f_s$ and $g_1, \dots, g_t$ are bases of the same ideal in*

Once a monomial order $>$ has been chosen, one can then determine the leading term $\mathrm{LT}(f)$ in each polynomial f, and order any two monomials. This in turn allows one to introduce the division algorithm

Theorem 2.3 (General Division Algorithm). *Fix a monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, and let $F = (f_1, \dots, f_s)$ be an ordered s-tuple of polynomials. Then every f $k[x_1, \dots, x_n]$ can be written as*

$$f = a_1 f_1 + \cdots a_s f_s + r, \tag{2.2}$$

where a_i, r $k[x_1, \dots, x_n]$ and either $r = 0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any of $\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_s)$. We call r a **remainder** *of f on division by F. Furthermore, if $a_i f_i = 0$, then we have $\mathrm{multideg}(f) \geq \mathrm{multideg}(a_i f_i)$.*

Remark 2.4. The remainder r in (2.2) (and the quotient monomials a_i), is not unique as it depends on the monomial order and on the division order of f by the polynomials in F. This last shortcoming of the Division Algorithm disappears when we divide polynomials by a Gröbner basis.

Remark 2.5. The termination of the Division Algorithm in $k[x_1, \dots, x_n]$ is guaranteed by the fact that $k[x_1, \dots, x_n]$ is a noetherian ring. For the actual algorithm, see for example [19] or [21].

The next concept needed is that of a monomial ideal so that we can state

Now that we know that every ideal in $k[x_1, \ldots, x_n]$ is finitely generated, we are ready to define a Gröbner basis for an ideal I $k[x_1, \ldots, x_n]$.

Definition 2.10. Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a **Gröbner basis** for I if

$$\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t)\ n$$ [illegible]

Example 2. Let $f_1 = x^4 - 3xy, f_2 = x^2y - 2 \in k[x, y]$ and *lex* order with $x > y$. Then, $\mathrm{LT}(f_1) = x^4$, $\mathrm{LT}(f_2) = x^2y$ and

$$S(f_1, f_2) = \frac{x^4y}{x^4} \cdot f_1 - \frac{x^4y}{x^2y} \cdot f_2 = y \cdot f_1 - x^2 \cdot f_2 = -3xy^2 + 2x^2 \in \langle f_1, f_2 \rangle.$$

Since $\mathrm{LT}(S(f_1, f_2))$ divisible by neither $\mathrm{LT}(f_1)$ nor $\mathrm{LT}(f_2)$, or, $\mathrm{LT}(S(f_1, f_2)) \notin \langle \mathrm{LT}(f_1), \mathrm{LT}(f_2) \rangle$, we see that f_1, f_2 is *not* a Gröbner basis of $\langle f_1, f_2 \rangle$.

Buchberger's algorithm for finding a Gröbner basis can be described as follows:

Buchberger's Algorithm. *Given* $\{f_1, \ldots, f_s\} \subset k[x_1, \ldots, x_n]$, *consider the algorithm which starts with* $F = \{f_1, \ldots, f_s\}$ *and then repeats the two steps*

- *(Compute Step) Compute* $\overline{S(f_i, f_j)}^F$ *for all* $f_i, f_j \in F$ *with* $i < j$,
- *(Augment step) Augment* F *by adding the non-zero* $\overline{S(f_i, f_j)}^F$ *until the Compute Step gives only zero remainders. The algorithm always terminates and the final value of* F *is a Gröbner basis of* $\langle f_1, \ldots, f_s \rangle$.

We will see later that all of the above steps from defining a monomial order through defining a Gröbner basis, S-polynomials, and a new algorithm in the non-commutative cases of interest to us – Grassmann and Clifford algebras – will be in principle repeated with certain modifications that will need to account for non-commutativity of these algebras and for the fact that, in general, these algebras unlike $k[x_1, \ldots, x_n]$ are not domains.

Example 3. Let $F_1 = \{f_1, f_2\}$ where $f_1 = 4(x_1 - 1)^2 + 4x_2^2 + 4x_3^2 - 9$ and $f_2 = (x_1 + 1)^2 + x_2^2 + x_3^2 - 4$ are as in Example 1. For the monomial order *lex* order with $x_1 > x_2 > x_3$, we find $f_3 = \overline{S(f_1, f_2)}^{F_1} = -16x_1 + 7$, so we extend F_1 to $F_2 = \{f_1, f_2, f_3\}$. Then, $f_4 = \overline{S(f_1, f_3)}^{F_2} = 495 - 256x_2^2 - 256x_3^2$, so we extend F_2 to $F_3 = \{f_1, f_2, f_3, f_4\}$. Next we find that $\overline{S(f_1, f_4)}^{F_3} = 0$. Thus, we have

$$\overline{S(f_1, f_2)}^{F_3} = \overline{S(f_1, f_3)}^{F_3} = \overline{S(f_1, f_4)}^{F_3} = 0.$$

Furthermore, we find that $\overline{S(f_2, f_3)}^{F_3} = \overline{S(f_2, f_4)}^{F_3} = \overline{S(f_3, f_4)}^{F_3} = 0$. Since $\overline{S(f_i, f_j)}^{F_3} = 0$ for all $i < j$ and $f_i, f_j \in F_3$, we conclude that a Gröbner basis for $I = \langle f_1, f_2 \rangle$ finally is

$$\begin{aligned} F_3 = \{&4(x_1 - 1)^2 + 4x_2^2 + 4x_3^2 - 9,\ (x_1 + 1)^2 + x_2^2 + x_3^2 - 4, \\ &- 16x_1 + 7, 495 - 256x_2^2 - 256x_3^2\}. \end{aligned} \tag{2.5}$$

Before we show specific computational examples of applying Gröbner bases in polynomial rings, we need to make the following observations:

- Automatic geometric theorem proving [15, 19].
- Expressing invariants of a finite group, e.g., symmetric polynomials, in terms of generating invariants [19, 52].
- Finding relations between polynomial functions, e.g., interpolating functions (syzygy relations)[2]

the hyperboloid $4x_1^2 + 4x_2^2 - 4x_{12}^2 = 1$. The primitive idempotents $\frac{1}{2}(1 \pm \mathbf{e}_1)$ and $\frac{1}{2}(1 \pm \mathbf{e}_2)$ belong to this variety when $x_{12} = x_2 = 0$ and $x_{12} = x_1 = 0$, respectively. For a classification of families of general idempotents in Cli�ord algebras see [6].

Our second example is related to the screw theory represented in the language of Cli�ord algebra $C_{0,3,1}$. This algebra contains a copy of the group of rigid motions $SE(3)$, its Lie algebra, the screws, and elements corresponding to points, lines and planes in Euclidean space R^3. [48] In fact, in [49], Selig and Bayro-Corrochano take two copies of that algebra and use the Cli�ord algebra $C_{0,6,2}$ to study momenta and inertia. Inpond‘

polynomials defined by the relations (3.5) and then by reducing all coefficients of the product $(g\bar{g})Q_0(g\bar{g})$ modulo G. Since we are reducing modulo the Gröbner basis, remainders of the reduction are uniquely defined. The Gröbner basis G for the $lex(\alpha_0 > \alpha_1 > \alpha_2 > \alpha_3 > \beta_0 > \beta_1 > \beta_2 > \beta_3)$ contains four polynomials including the original two polynomials. Computing the difference we find

$$
\begin{aligned}
(g\bar{g})Q_0(g\bar{g}) - Q_0 = {} & h_1 e_1 e_2 a_3 a + h_2 e_1 e_3 a_2 a + h_3 e_1 e a_1 a \\
& + h_4 e_1 e a_2 a_3 + h_5 e_2 e_3 a_1 a + h_6 e_2 e a_1 a_3 \\
& + h_7 e_2 e a_2 a + h_8 e_3 e a_1 a_2 + h_9 e_3 e a_3 a
\end{aligned}
\tag{3.10}
$$

where $h_j \quad R[\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3]$ and $\overline{h_j}^{G} = 0$ for $j = 1, \ldots, 9$. Thus, indeed, Q_0 is invariant under the group action $(g\bar{g})Q_0(g\bar{g})$ where g is the rigid transformation. The same way one can show that Q_0 is not invariant under the action of g or $\bar{g}$ alone.

In general, the action on P shown in (3.7) needs to be computed modulo G as well. Later in their paper Selig and Bayro-Corrochano deduce that the inertia N must transform according to $N \to (g\bar{g})N(g\bar{g})$ and hand-compute such transformation of N when $g = 1 + \frac{1}{2}t_x e_1 e$. When g is more general, or as general as possible, hand computation is no longer practical and the above approach is superior.

Finally, we show a simple add-on procedure to CLIFFORD/Bigebra [3] that can reduce symbolic polynomial coefficients of any element in the defined Clifford algebra modulo a set of polynomial relations, e.g., as in (3.5). This approach is particularly useful when computing action of the Lipschitz group or the spin groups [40] modulo relations that coefficients of general elements of these groups must satisfy.

```
ReduceClipolynom:=proc(p::{cliscalar,clibasmon,climon,clipolynom})
                  local F,tmon,T,C,i,m,G:
F,tmon:=op(procname):
if type(p,clibasmon) then return p end if:
G:=Groebner:-Basis(F,tmon);
if type(p,cliscalar) then return Reduce(p,G,tmon) end if;
T:=convert(cliterms(p),list):
C:=[seq(coeff(p,m),m=T)];
C:=map(Groebner:-Reduce,C,G,tmon);
```

For our third example, we need the following result [19].[3]

Proposition 3.1. *Suppose that $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ are given. Fix a monomial order $k[x_1, \ldots, x_n, y_1, \ldots, x_m]$ where any monomial involving one of $x_1, \ldots, x_n$ is greater than all monomials in $k[y_1, \ldots, y_m]$. Let G be a Gröbner basis of the ideal $J = \langle f_1 - y_1, \ldots, f_m - y_m \rangle \subset k[x_1, \ldots, x_n, y_1, \ldots, x_m]$. Given $f \in k[x_1, \ldots, x_n]$, let $g = \overline{f}^G$ be the remainder of f on division by G. Then*

(i) *$f \in k[f_1, \ldots, f_m]$ if and only if $g \in k[y_1, \ldots, y_m]$.*
(ii) *If $f \in k[f_1, \ldots, f_m]$, then $f = g(f_1, \ldots, f_m)$ is an expression of f as a polynomial in $f_1, \ldots, f_m$.*

Example 7 (Symmetric polynomials). Let G be the symmetric group S_3. Let

$$\sigma_1 = x_1 + x_2 + x_3, \quad \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad \text{and} \quad \sigma_3 = x_1x_2x_3$$

be the *elementary symmetric polynomials* in x_1, x_2, x_3. [52] A Gröbner basis F for the ideal $I = \langle \sigma_1 - y_1, \sigma_2 - y_2, \sigma_3 - y_3 \rangle$ in $lex(x_1, x_2, x_3, y_1, y_2, y_3)$ order is

$$F = [x_3^3 - x_3^2y_1 + y_2x_3 - y_3,\ x_2^2 + x_2x_3 - x_2y_1 + x_3^2 - x_3y_1 + y_2,\ x_1 + x_2 + x_3 - y_1]$$

Let

$$f = x_1^2x_2 + x_1x_2^2 + 3x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 - x_1^2x_2^2x_3^2.$$

It can be checked directly that $f(\mathbf{x}) = f(\sigma\mathbf{x})$, $\forall \sigma \in S_3$. That is, f is invariant under S_3 and $f \in k[x_1, x_2, x_3]^{S_3}$. Reducing f modulo F gives $g = \overline{f}^F = y_1y_2 - y_3^2 \in k[y_1, y_2, y_3]$. Thus, by part (i) of the above Proposition, we see again that f is symmetric. Furthermore, from part (ii) we get that $f = \sigma_1\sigma_2 - \sigma_3^2$.

For more examples on finite group generators and finding the so called *syzygy relations* (or, *syzygies*), see [19], [52]). For a small Maple package related to finite group invariants as well as generators (relations) of syzygy ideals, see SP package. [5]

4. PBW rings and algebras

There is a natural and important progression in developing the theory of Gröbner bases for Grassmann and Clifford algebras through the so called **left Poincaré-Birkhoff-Witt (PBW) rings and algebras.** While these rings are non-commutative, they possess a monomial basis and an admissible order can be defined on standard monomials. Furthermore, like ordinary polynomial rings $k[x_1, \ldots, x_n]$, they are domains and are left noetherian (Hilbert's Basis Theorem). Furthermore, PBW rings have the terminating multivariable division algorithm property, and every non-zero left ideal in a PBW ring possesses a Gröbner basis. In particular, if G is a Gröbner basis for a non-zero left ideal I in a PBW ring R, any "polynomial"

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[3] Procedure `isContained` from SP

where $T(\mathfrak{g})$ is the tensor algebra over the linear space of $\mathfrak{g}$ and I is a two-sided ideal generated by $x \quad y - y \quad x - [x, y], \quad x, y \quad \mathfrak{g}$. Therefore, $U(\mathfrak{g})$ is a PBW algebra and

$$U(\mathfrak{g}) = k\{x_1, \ldots, x_n;\ x_i x_j = x_j x_i + [x_j, x_i], \quad _{deglex}\}$$

- Let $\mathbf{q}$ be a multiplicatively anti-symmetric $n \times n$ matrix over k, i.e., $q_{i,j} = 0$ and $q_{i,j} = q_{j,i}^{-1}$ for all $1 \leq i, j \leq n$. The (multiparameter) n-dimensional quantum space $k_{\mathbf{q}}[x_1, \ldots, x_n]$ associated to $\mathbf{q}$ is the quotient of the free k-algebra $k \ x_1, \ldots, x_n$ by the two-sided ideal associated to the relations $Q = \{x_j x_i = q_{ji} x_i x_j,\ j > i\}$. Let be any admissible order on $\mathbb{N}^n$. Then

$$O_{\mathbf{q}}(k^n) = k\{x_1, \ldots, x_n;\ Q, \quad \}$$

is a PBW algebra.
- There are constructive methods to obtain new (left) PBW rings as Ore extensions of a given (left) PBW ring. For example, skew polynomial Ore algebras and rings of di□erent3.

for□Td[(x)]TJ-JF7□6.9738089□0□Td[(8rb438(i551□Td[(and)-438(ri)1(ngs)-439(of)-4344028(ew)-299(p)-27(od[(h)]eew)-29

A k-algebra $A = T/I = k\langle x_1, x_2, \ldots, x_n \mid x_j x_i = c_{ij} x_i x_j + d_{ij},\ 1 \le i < j \le n\rangle$ is called a G-algebra in n variables, if the following conditions hold:

- Ordering condition

(when $r = 1$ or a vector of polynomials otherwise) of f and g as:

$$\text{LeftSpoly}(f, g) = \begin{cases} x^{\gamma-\alpha} f - \dfrac{\text{LC}(x^{\gamma-\alpha} f)}{\text{LC}(x^{\gamma-\beta} g)} x^{\gamma-\beta} g, & \text{if } i = j, \\ 0, & \text{if } i = j. \end{cases}$$

The LeftSpoly form is needed for the Gröbner basis algorithm. A characterization of a Gröbner basis within a G-algebra can now be given. It will be the foundation for implementing a Gröbner basis algorithm.

Theorem 5.9. *Let $I \subseteq A^r$ be a left submodule and $G = \{g_1, \ldots, g_s\} \subseteq I$ and let* $\text{LeftNF}(-|G)$ *be a left normal form on A^r with respect to G.*

6. Grassmann and Clifford algebras in Plural

G-algebras are defined in Plural [50] using `ring` command extended to non-commutative variables. Then, a *GR*-algebra is defined as a quotient of a *G*-algebra modulo a two-sided ideal *I*. It is of the type `qring`, for example, `qring Q = twostd(I)`

Example 9. Consider an ideal $I = \langle 2\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_2 - 4\mathbf{e}_3 \wedge \mathbf{e}_4, \mathbf{e}_1 \rangle \subset \bigwedge \mathbb{R}^4$. Plural and TNB return the following Gröbner basis for I in the `Deg[Lex]` order:

$$\{\mathbf{e}_2 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_4,\ 4\mathbf{e}_3 \wedge \mathbf{e}_4 - \mathbf{e}_2,\ \mathbf{e}_1\} \tag{6.1}$$

Example 10. Consider polynomials $f_1 = \mathbf{e}_5 \wedge \mathbf{e}_6 - \mathbf{e}_2 \wedge \mathbf{e}_3$ and $f_2 = \mathbf{e}_4 \wedge \mathbf{e}_5 - \mathbf{e}_1 \wedge \mathbf{e}_3$ in $\bigwedge \mathbb{R}^6$. The Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$ in `Deg[Lex]` order returned by Plural and TNB is

$$\{\mathbf{e}_{145},\ \mathbf{e}_{245} + \mathbf{e}_{156},\ \mathbf{e}_{256},\ \mathbf{e}_{345},\ \mathbf{e}_{356},\ \mathbf{e}_{13} - \mathbf{e}_{45},\ \mathbf{e}_{23} - \mathbf{e}_{56}\} \tag{6.2}$$

where $\mathbf{e}_{145} = \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5$, etc. This basis is different from the Gröbner GLB basis in Stokes (see below) for this ideal which is

$$\{\mathbf{e}_{56} - \mathbf{e}_{23},\ \mathbf{e}_{45} - \mathbf{e}_{13},\ \mathbf{e}_{234} + \mathbf{e}_{136},\ \mathbf{e}_{1236}\}. \tag{6.3}$$

Basis (6.2) is a GLIB basis in Stokes' terminology that solves the ideal membership problem while basis (6.3) is a GLB basis that does not solve that problem, hence it is different from (6.2).

Example 11. We compute a Gröbner basis in a left ideal $I = \langle \mathbf{e}_1 + 2\mathbf{e}_2, 3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2 \rangle$ in $C\ell_{2,0}$. The monomial order is `dp= Deg[Lex]`.

```
LIB "clifford.lib";
ring R = 0,(e1,e2),dp;
option(redSB);
option(redTail);
matrix M[2][2];
M[1,1]=2;M[2,2]=2;
clifAlgebra(M);
qring Q =twostd(clQuot);
ideal I =
e1
+ 2*e2
,
3*e1
+ e1*e2
;
short=0;
ideal GB = std(I);
```

The Gröbner basis for I is $\{1\}$, hence the ideal I is the entire algebra.

Example 12. Take $C\ell_{2,0} = \mathrm{Mat}(2, \mathbb{R})$ and a primitive idempotent $f = \frac{1}{2}(1 + \mathbf{e}_1)$. Let $S = C\ell_{2,0} f = \mathrm{span}_{\mathbb{R}}\{f, \mathbf{e}_2 f\}$ be a spinor ideal. Then a Gröbner basis for S in the monomial order `dp = Deg[Lex]` is $h = 1 + \mathbf{e}_1$. Note that $h = 2f$ is an almost idempotent.

Example 13. Consider $C\ell_{3,3} = \mathrm{Mat}_8(\mathbb{R})$ with a monomial order `Deg[InvLex]`. This order has no special name in Plural but we'll call it "degree inverse lex order" drp. It can be entered in Plural as (a(1:n),rp) where n refers to the number of non-commuting generators.

Algebra C

This last example shows the usefulness of the Gröbner basis for any left spinor ideal $S \quad C_{p,q}$: Element $p \quad C_{p,q}$ belongs to S if and only if NF paravectors a, b R $R^{0,7}$ as shown by Lounesto. Element $\mathbf{v}$ is uniquely defined by the chosen primitive idempotent f as $\mathbf{v} = \mathbf{w}\mathbf{e}_{12\ldots7}$ and $\mathbf{w} = 8(f + \hat{f}) - 1$. [5] Here, $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713}$. A Gröbner basis for the ideal $S = C_{0,7}f$ in monomial order `drp = Deg[InvLex]` is:

$$g_1 = \mathbf{e}_7 f,\ g_2 = \mathbf{e}_6 f,\ g_3 = \mathbf{e}_5 f,\ g_4 = \mathbf{e}_4 f,\ g_5 = \mathbf{e}_3 f,\ g_6 = \mathbf{e}_2 f,\ g_7 = \mathbf{e}_1 f. \quad (6.14)$$

It can be easily checked that $g_i g_j = 0$, $i, j = 1, \ldots, 7$. Units $\mathbf{e}_i$, $i = 1, \ldots, 7$, link these generators to the idempotent f. Of course, a Gröbner basis for the corresponding ideal $\hat{S} = C_{0,7}\hat{f}$ is

$$\hat{g}_1 = \mathbf{e}_7 \hat{f},\ \hat{g}_2 = \mathbf{e}_6 \hat{f},\ \hat{g}_3 = \mathbf{e}_5 \hat{f},\ \hat{g}_4 = \mathbf{e}_4 \hat{f},\ \hat{g}_5 = \mathbf{e}_3 \hat{f},\ \hat{g}_6 = \mathbf{e}_2 \hat{f},\ \hat{g}_7 = \mathbf{e}_1 \hat{f}, \quad (6.15)$$

and again $\hat{g}_i \hat{g}_j = 0$. It is interesting to note that all seven nilpotent polynomials (6.14) together with the idempotent f constitute eight basis elements for S considered as a vector space.

In our last example in this section we will list Gröbner bases G for left spinor ideals $S = C_{p,q}f$ in all semisimple and simple Clifford algebras in dimensions $2 \leq p + q \leq 8$. Furthermore, we discuss only Clifford algebras $C_{p,q}$ such that $fC_{p,q}f = R$, that is, when $p - q$

- (Exception) $S = C\ell_{0,6}f = \mathrm{LI}(g_1, \ldots, g_5)$ where $f = u_i g_i$, $g_i g_1 = 0$, $g_i g_j = 0$ for.]TJF

7. GLB and GLIB bases in Grassmann algebras

In 1990 Timothy Stokes showed that Grassmann algebra is suitable for algorithmic treatment when treated as graded-commutative algebra of “exterior polynomials”. In [51] he defined two different Gröbner bases for left ideals in (super) Grassmann polynomial algebra $_{n,m}$ of order (n, m) over a field k, char k = 2. Furthermore,

Proposition 7.3 (Stokes).

Theorem 7.7 (GLB Characterization Theorem). *The following conditions are equivalent.*

(i) *F is a GLB.*

(ii) *If $f_1, f_2 \quad F$, and $t \quad T_{n,m}$ satisfies $t \cdot \mathrm{lcm}(\mathrm{LMon}(f_1), \mathrm{LMon}(f_2)) = 0$, then $t \cdot S(f_1, f_2) \to_F 0$ where $S(f_1, f_2)$ is an S-polynomial of f_1, f_2.*[8]

(i) *F is a GLIB.*
(ii) *If $f_1, f_2 \quad F$, then for any $t \quad T_{n,m}$, $t \cdot f_1 \rightarrow_F 0$ and $t \cdot \mathrm{LeftSpoly}(f_1, f_2) \rightarrow_F 0$.*
(iii) *For $f_1, f_2 \quad F$ and for any $t_1, t_2 \quad T_{n,m}$ satisfying the conditions*

$$t_1 \cdot \mathrm{LTerm}(f_1) = 0 \text{ and } t_2 \cdot \mathrm{lcm}(\mathrm{LTerm}(f_1), \mathrm{LTerm}(f_2)) = 0,$$

then $t_1 \cdot f_1 \rightarrow_F 0$ and $t_2 \cdot \mathrm{LeftSpoly}(f_1, f_2) \rightarrow_F 0$.

Remark 7.10. See Stokes [51] for a discussion of correctness and termination of his algorithm to compute a GLIB basis G from the initial list F. Stokes proves that $\mathrm{LI}(F) = \mathrm{LI}(G)$.

Example 19. Let f_1, f_2, f_3 be as in Example 18. A GLB basis for $\mathrm{LI}(f_1, f_2, f_3)$ was shown in (7.1). A GLIB basis G_1 for $\mathrm{LI}(f_1, f_2, f_3)$ for the same monomial order `Deg[InvLex]` required computation of six S-polynomials and is given by

$$g_1 = \mathbf{e}_{2456} - \mathbf{e}_3, \quad g_2 = \mathbf{e}_{14}$$

- When reducing an S-polynomial $S(f_i, f_j) \in R = k[x_1, \ldots, x_n]$ modulo a finite set of polynomials F, for example, when computing a Gröbner basis, suppose $\overline{S(f_i, f_j)}^F = 0$. Then, $\overline{m \cdot S(f_i, f_j)}^F = 0$ for any monomial $m = x^\alpha \in R$. This is often not the case in Grassmann or Clifford algebra due to the presence of non-zero zero divisors. This complicates computation of Gröbner bases in these algebras.

A major difference aside from the non-commutativity when computing Gröbner bases in Grassmann and Clifford algebras is the presence, if not abundance, of non-zero zero divisors. Algorithms to compute a left normal form and then a left Gröbner basis in [35, 36] generalize the classical Buchberger's algorithm from $k[x_1, \ldots, x_n]$ to a quotient GR-algebras and solve the ideal membership problem. However, care must be taken as vanishing of an S-polynomial modulo a set of "polynomials" in Grassmann or Clifford algebra does not guarantee its vanishing when the S-polynomial is pre-multiplied by a basis monomial. This has been shown clearly by Stokes [51] who has introduced two types of Gröbner bases in left ideals in Grassmann algebra: a GLB basis which guarantees uniqueness of a the remainder, and GLIB which also guarantees that $\overline{f}^G = 0$ modulo a GLIB-type Gröbner basis G is equivalent to $f \in \langle G \rangle$. See [11] for implementation of GLB and GLIB bases for Grassmann algebras in a Maple package `TNB`.

Finally, we mention that non-commutative Gröbner bases in Grassmann algebras and the issue of ideal membership surface when analyzing systems of partial differential equations that arise in physics, i.e., in exterior differential systems as shown in [29] and references therein. In particular, Hartley and Tuckey provide another approach through the so called *saturating sets* to Gröbner bases in Grassmann and Clifford algebras in a REDUCE package called `XIDEAL`. A major application emphasized in the paper is that Gröbner bases may help simplify exterior differential systems and so help solve systems of partial differential equations.

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Acknowledgments

The author thanks Viktor Levandovskyy (RWTH Aachen University) for showing him how to use library `clifford.lib` for Plural [37]. Thanks are also due to Hans Schönemann (University of Kaiserslautern) for his help with monomial orders in Plural. Finally, the author expresses his gratitude to Bertfried Fauser (Universität Konstanz) for his helpful comments when preparing this paper and, in particular, for suggesting a possible connection between Gröbner bases in spinor ideals in $C\ell_{0,7}$ and triality [24].

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