OVERCOMING STUDENTS' DIFFICULTIES IN LEARNING TO UNDERSTAND AND CONSTRUCT PROOFS[1]

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When a topologist colleague was asked to teach remedial geometry, he used the process of learning to construct proofs may even involve students coming to know themselves better. Indeed, the above student's comment, about waking up with a math problem, suggests that she has learned to persist until she eventually comes up with a solution, even if that's in the middle of the night. Unfortunately, many students believe that they either know how to solve a problem (prove a theorem) or they don't, and thus, if they don't make progress within a few minutes, they give up and go on to something else.

Of course, undergraduate students do not learn to construct proofs only in transition-to-proof courses. They tend to improve their ability to construct proofs throughout the entire undergraduate mathematics program. Some departments do not even offer transition-to-proof courses, and some combine them with mathematics content courses such as discrete structures. Occasionally, students are offered an R. L. Moore type course,[4] that is, a course in which the textbook and lectures are replaced by a brief set of notes and in which the students produce all the proofs. To some extent, the emphasis in such courses is on a deep understanding of the mathematical content -- however, it has been our experience that once students get started in such courses they often improve their proof making abilities very rapidly. Unfortunately, a few students may have great difficulty getting started.

In whatever setting students are to progress in their proving abilities, one might expect the teaching to be somewhat special. In many university mathematics content courses, teachers can profitably explain mathematical theorems and why they are true, but in teaching the skills and problem solving abilities involved in proving, one should also expect to emphasize guiding students' practice. In developing such teaching, it can be useful to ask: What kinds of difficulties do student have, and how might these difficulties be alleviated?

We will describe some results from the mathematics education research literature that address these questions. However, it is important to note that this research typically makes no claim (as one familiar with other social sciences might expect) that all, or even

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And, to a scientist, it can mean the positive results of an empirical investigation. To comprehend the special way that "proof" is used in mathematics can take time and such everyday meanings can get in the way.

Views of High School Geometry Students

A number of studies have documented the finding that the concept of mathematical proof is not quickly or easily grasped. For example, in the middle of a year-long U.S. high school geometry course, after being introduced to deductive proof, students in five classes were given a short instructional unit designed to highlight differences between measurement of examples and deductive proof. Seventeen of the students were interviewed and asked to compare and contrast two arguments (for different theorems) -- a deductive proof and an argument containing four examples using differently shaped triangles. Some of these students had a nuanced "evidence is proof" view. They considered empirical evidence to be sufficient proof for a statement about all triangles, provided one took measurements of each type of triangle -- acute, obtuse, right, scalene, equilateral, and isosceles. Others had a qualified view of deductive proof, believing that a two-column proof only proved a theorem for the type of triangle depicted in the accompanying figure and would need to be reproved, perhaps using the same steps, for other types of triangles. Most surprising and quite disturbing, especially after an natural outgrowth of testing, refining, and verifying their own conjectures, the results were disappointing, even disastrous, for students entering England's universities. Apparently, the verifications, that had been intended to be student constructed deductive arguments, were instead turned into standardized templates and empirical arguments (Coe & Ruthven, 1994).

By 1995, the situation had caused so much concern that the London Mathematical Society issued a report on the problems as mathematicians perceived them. The report stated that recent changes in school mathematics "have greatly disadvantaged those who need to continue their mathematical training beyond school level." In particular, the following problems were cited: "serious lack of essential technical facility -- the ability to undertake numerical and algebraic calculation with fluency and accuracy," "a marked decline in analytical powers when faced with simple problems requiring more than one step," and "a changed perception of what mathematics is -- in particular of the essential place within it of precision and proof" (London Mathematical Society, 1995, p. 2).

After the public outcry of mathematicians, a large-scale study, called *Justifying and Proving in School Mathematics*, was undertaken. The study surveyed 2,459 high-attaining Year 10 students (14-15 years old, that is, comparable to U.S. high school sophomores) in 94 classes from 90 English and Welsh schools. In a series of papers and reports, it was convincingly documented that it was the new National Curriculum, as implemented by teachers, that was, in large part, responsible for the perceived decline in U.K. students' notions of proof and proving (Hoyles, 1997; Healy & Hoyles, 1998, 2000).

What did this large, mostly quantitative, but partly qualitative, study find? In the Executive Summary of the report (Healy & Hoyles, 1998), one finds the following conclusions, amongst others. (1) Students' performance on constructing proofs was "very disappointing." These better-than-average[6] students were asked to judge whether a number of empirical, narrative, and algebr.1483r1.15 T 0 TT6.96& Hoy0 Tya

> highest-attaining students may simply be explained by their lack of familiarity with the process of proving. (Healy & Hoyles, 1998, p. 6)

Thus, the way a curriculum conveys proof and proving is clearly crucial, but skilled and knowledgeable teachers are also critical for implementing such a curriculum. The current *NCTM Standards* argument demonstrating the converse of the statement, *If* $x > 0$, *then* $x + \frac{1}{x} \geq 2$, to be a proof of it; these teachers seemed to focus on the correctness of the algebraic manipulations, rather than on the validity of the argument (Knuth, 2000b).

Given this result regarding some better and more committed secondary mathematics teachers, can one expect that beginning U.S. university students would be reasonably skilled at proof and proving? Would they, for example, understand the distinction between proof and empirical argument? Probably not.

University Students' Views of Proof

Undergraduate students sometimes come to see proofs and proving as unrelated to their own ways of thinking. In order to cope, they may employ mimicking strategies with the result that they develop various views of proof that are unusual from a mathematician's viewpoint; Harel and Sowder (1998) have classified some of these "proof schemes." These are *not* techniques of (mathematical) proof, but rather kinds of arguments, sometimes incorrect or incomplete, that some university students find convincing, and may even think of as proofs.[8] An example of preservice elementary teachers' views of proof follows.

In the 10th week of a sophomore-level mathematics course, 101 preservice elementary teachers were asked to judge verifications of a familiar result, *if the sum of the digits of a whole number is divisible by 3, then the number is divisible by 3,* and an unfamiliar result, provided one avoids the pitfall, described above, of allowing mathematical "investigations" to conclude with purely empirical justifications.

Understanding and Using Definitions and Theorems

Not only are there everyday uses of "proof" that might compound students' difficulties in coming to know what a mathematical proof is, students can be confused about the role of definitions in mathematics.

Mathematical Definitions

Everyday descriptive, or dictionary, definitions[10] describe both concrete and abstract things, already existing in the world, such as trees, love, democracy, or epistemology. They can be both redundant and incomplete, and it is never clear whether all aspects of a definition must apply for its proper use. In contrast, mathematical definitions[11] bring concepts into existence; the concept, say of group, means nothing more and nothing less than whatever the definition says. While all parts of a mathematical definition definitely need to be considered when producing examples and nonexamples, other features of prospective examples need not be considered. This point is often missed. When asked whether $F = 151 \times 157$ is prime, a number of preservice elementary teachers correctly, but irrelevantly noted that both 151 and 157 are prime, before going on to conclude that their product

2, 3, 5, 7, 9, 11, 15, or 63, a majority (29 of 54) stated that 3, 5, 7 were divisors since those were among the factors in the prime decomposition. However, sixteen were unable to apply similar reasoning to 2 and 11, some noting instead that “M is an odd number” so “2 can’t go into it” or resorting to calculations (like the above) for 11. In addition, many of these students believed that prime decomposition means decomposition into *small* primes (see also Zazkis & Liljedahl, 2004).

Undergraduate students often ignore relevant hypotheses or apply the converse when it does not hold. A well-know instance is the use, by Calculus II students, of the converse of: *If* $\sum a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$, as an easy, but incorrect, test for convergence. Some calculus books go on to point out that this theorem provides a Test for Divergence. But, perhaps it would be better to explicitly state the contrapositive, *If* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum a_n$ *diverges.*

Sometimes undergraduate students use theorems, especially theorems with names, as vague "slogans" that can be easily retrieved from memory, especially when they are asked to answer questions to which the theorems seemingly apply. For example, Hazzan and Leron (1996) asked twenty-three abstract algebra students: *True or false? Please justify your answer. "In* S_7 *there is no element of order 8."* It was expected that students would check whether there was a permutation in S_7 having 8 as the least common multiple of the lengths of its cycles. Instead, 12 of the 16 students who gave incorrect answers invoked Lagrange's Theorem[12] or its converse. Seven of them incorrectly invoked Lagrange's Theorem to say the statement was false -- there is such an element since 8 divides 5040. Another two students inappropriately invoked a contrapositive form of Lagrange's Theorem to say the statement was true because 8 doesn't divide 7. The authors go on to point out that students often think Lagrange's Theorem is an existence theorem, although its contrapositive shows that it is a non-existence theorem: *If k doesn't divide o*(*G*)*, then there doesn't exist a subgroup of order k.* Perhaps it would be good to state this version explicitly for students.

The above examples refer to students’ misuse of theorems when they are asked to solve specific problems, for example, determine whether a number is prime or a series converges, or decide whether a group has an element of order 8. However, it is not hard to imagine similar difficulties when students attempt to use theorems in constructing their own proofs.

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We describe the reasoning progress of Stephanie, one of the children with whom Maher and Martino (1996a, 1996b, 1997) began their long-range, but occasional, interventions commencing in Grade 1. By Grade 3, the children had begun building physical models and justifying their solutions to the following problem: How many different towers of heights 3, 4, or 5 can be made using red and yellow blocks? Stephanie not only justified her solutions, she validated or rejected

> her own ideas and the ideas of others on the basis of whether or not they made sense to her. . . . She recorded her tower arrangements first by drawing pictures of towers and placing a single letter on each cube to represent its color, and then by inventing a notation of letters to represent the color cubes. (Maher & Speiser, 1997, p. 174)

She used spontaneous heuristics like guess and check, looking for patterns, and thinking of a simpler problem, and developed arguments to support proposed parts of solutions, and extensions thereof, to build more complete solutions. Occasional interventions continued for Stephanie through Grade 7. Then in Grade 8 she moved to another community and another school and her mathematics was a conventional algebra course. The researchers interviewed her that year about the coefficients of $(a+b)^2$ and $(a+b)^3$. About the latter, she said "So there's a cubed . . . And there's three a squared b b cubed all but one student unsuccessfully began with the hypothesis -- $f \circ g$ is one-to-one -- rather than assuming that $g(x) = g(y)$. (Moore, 1994). They did not appear to know how to use the definition of one-to-one and relate that to the structure of their proofs.[13]

Unpacking the Logical Structure of Statements of Theorems

Another difficulty students have when constructing their own proofs is an inability to unpack the logical structure of informally stated theorems -- theorems that depart from a natural language version of predicate calculus. That is, theorems that omit specific mention of some variables or depart from the use of *for all, there exists, and, or, not, if-then*, and *if-and-only-if* in a significant way. For example the statement, *Differentiable functions are continuous*, is informal because a universal quantifier and the associated variable are understood by convention, but not explicitly indicated. Similarly, *A function is continuous whenever it is differentiable* is informal because it departs from the familiar *if-then* expression of the conditional as well as not explicitly specifying the universal quantifier and variable.

Being able to unpack the logical structure of such informally stated theorems is important because the logical structure of a mathematical statement is closely linked to the overall structure of its proof. For example, knowing the logical structure of a statement helps one recognize how one might begin and end a direct proof of it. When asked to unpack the logical structure of four informally worded syntactically correct statements, two true and two false, undergraduate mathematics students, many in their third or fourth year, did so correctly just 8.5% of the time. Especially difficult for them was the correct interpretation of the order of the existential and universal quantifiers in the false statement: *For* $a < b$*, there is a* c *so that* $f(c) = y$ *whenever* $f(a) < y$ *and* $y < f(b)$[14] (Selden & Selden, 1995).

Furthermore, the ability to unpack the logical structure of the statement of a theorem also allows one to know whether an argument proves that statement, as opposed to some other statement. For example, eight mid-level undergraduate mathematics and mathematics education majors were asked to judge the correctness of student-generated "proofs" of a single theorem.[15] Upon finding a proof of the converse particularly easy to follow, four initially incorrectly stated that it was a proof of the original statement, and two of these maintained this view throughout the interview (Selden & Selden, 2003).

Understanding the Effect of Existential and Universal Quantifiers

One source of students' difficulties in discerning the logical structure of theorems is a lack of understanding of the meaning of quantifiers and that their order matters.

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[13] In a rather formally written proof, one might begin something like, "Suppose $f \circ g$ is one-to-one." But (with this definition), the hypothesis is not used until one attempts to prove that g is one-to-one by assuming $g(x) = g(y)$. An alternative definition, $x \neq y$ implies $f(x) \neq f(y)$, might have made this particular theorem easier to prove, but apparently the students did not think of using it.

[14] If f

Undergraduate students often consider the effect of an interchange of existential and universal quantifiers as a mere rewording. For example, in another study, when given the two statements:

- *For every positive number a there exists a positive number b such that $b < a$.*
- *There exists a positive number b such that for every positive number a $b < a$.*

24 of 54 students in undergraduate mathematics courses, such as linear algebra and multivariable calculus, and 3 of 9 students in a beginning graduate abstract algebra course said they were "the same" or were mere

times, and perhaps, to the interviewer’s no longer accepting “unsure” as a response. Most of the errors detected were of a local/detailed nature rather than a global/structural nature, with only the two students who had proved the theorem themselves observing that the converse had been proved in (d).

When asked how they read proofs, the students said they attempted careful line-by-line checks to see whether each mathematical assertion followed from previous statements, checked to make sure the steps were logical, and looked to see whether any computations were left out. Several said they went through the proofs using an example. Also, for these students, a feeling of personal understanding or not--that is, of making sense or not--seemed to be an important criterion when making a judgment about correctness of a “proof.” Thus, what students say about how they read proofs seems a poor indicator of whether they can actually validate proofs with reasonable reliability. While these students tended to “talk a good line,” their judgments at Time 1 were no better than chance (46% correct).

On the other hand, even without explicit instruction, the reflection and reconsideration engendered by the interview process eventually yielded 81% correct judgments, suggesting that explicit instruction in validation could be effective (Selden & Selden, 2003). Indeed, several transition-to-proof textbooks include "proofs to grade,"[16]
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students actively try to enhance their concept images, for instance, by considering examples and nonexamples?

Getting to Know and Use a New Definition

In one study conducted by Dahlberg & Housman (1997), eleven students, all of whom had successfully completed introductory real analysis, abstract algebra, linear algebra, set theory, and foundations of analysis, were presented with the following formal definition. A function is called *fine* if it has a root (zero) at each integer. They were first asked to study the definition for five to ten minutes, saying or writing as much as possible of what they were thinking, after which they were asked to generate examples and nonexamples. Subsequently, they were given functions and asked to determine whether these were fine functions and, if so, why. Next, they were asked to determine the truth of especially difficult, in particular, for preservice teachers (Zazkis & Liljedahl, 2004). Similarly, irrational numbers have no such representation; thus, in proving results such as $\sqrt{2}$ is irrational or the sum of a rational and an irrational is irrational, one is led to consider proofs by contradiction -- something often difficult for beginning students.

Symbolic representations can make certain features transparent and others opaque.[18] For example, if one wants to prove a multiplicative property of complex numbers, it is often better to use the representation $re^{i\theta}$, rather than $x + iy$, and if one wants to prove certain results in linear algebra, it may be better to use linear transformations, *T*, rather than matrices. Students often lack the experience to know when a given representation is likely to be useful.

It has been argued that moving flexibly between representations (e.g., of functions given symbolically or as a graph) is an indication of the richness of a student's understanding of a concept (Even, 1998). Also, understanding an abstract mathematical concept can be regarded as possessing "a notationally rich web of representations and applications" (Kaput, 1991, p. 61).

Bringing Appropriate Knowledge to Mind

No one questions the need for content knowledge, sometimes referred to as resources,[19] To solve the above non-routine problem, one needs to know (1) that one might set the derivative of $4x^3 - x^4 - 30$ equal to zero to find its maximum -3 and (2) that solutions of the given equation are where this function crosses the x-axis (which it does not). Many of the students had these two resources, but apparently could not bring them to mind at an appropriate time. We conjectured that, in studying and doing homework, the students had mainly followed worked examples from their textbooks and had thus never needed to consider various different ways to attempt problems. Thus, they had no experience at bringing their assorted resources to mind. It seems very likely that a similar phenomenon could occur in attempting to prove theorems.

How does one think of bringing the appropriate knowledge to bear at the right time? To date, mathematics education research has had only a little to say about the difficult question of how an idea, formula, definition, or theorem comes to mind when it would be particularly helpful, and probably there are several ways. In their study of problem solving, Carlson and Bloom (2005) found that mathematicians frequently did not access the most useful information at the right time, suggesting how difficult it is to draw from even a vast reservoir of facts, concepts, and heuristics when attempting to solve a problem or to prove a theorem. Instead, the authors found that mathematicians' progress was dependent on their approach, that is, on such things as their ability to persist in making and testing various conjectures.

Our own personal experience of eventually bringing to mind resources that we had -- but did not at first think of using -- suggests that persistence, over a time considerably longer than that of the Carlson and Bloom interviews, can be beneficial. We conjecture that certain ideas get in the way of others, and that after a good deal of consideration, such unhelpful ideas become less prominent and no longer block more helpful ideas. This may be related to a psychological phenomenon that can take several forms; for example, in vision, if one fixates on a single spot in a picture, it will eventually disappear.

While coming to mind at the right time can be seen as an idiosyncratic, individual act, it may sometimes be related to the idea of transfer of one's knowledge. How does one come to see a new mathematical situation as similar to a previously encountered situation and bring the earlier resources to bear on the new situation?

Knowing What's Important and Useful

In addition to knowing what a proof is, being able to reason logically, unpack definitions, and apply theorems, and having a rich concept image of relevant ideas, one needs a "feel" for the content and what kinds of properties and theorems are important. Knowing what's important should go a long way towards bringing to mind appropriate resources.

Not Seeing that Geometry Theorems are Useful when Making Constructions

Seeing the relevance and usefulness of one's knowledge and bringing it to bear on a problem, or a proof, is not easy. Schoenfeld (1985, pp. 36-42) provides an example of two beginning college students who had completed a year of high school geometry and were asked to make a construction: *You are given two intersecting straight lines and a* *point P marked on one of them. Show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines.* During a 15-minute joint attempt, they made rough sketches and conjectures, and tested their conjectures by making constructions. When asked why their constructions ought or ought not to work, they responded in terms of the mechanics of construction, but did not provide any mathematical justification. Yet the next day they were able to give the proof of two relevant geometric theorems within five minutes. Apparently, these students simply did not see the relevance of these theorems at the time.

Knowing to Use Properties, Rather than the Definitions, to Check Whether Groups are Isomorphic

In another study, four undergraduates who had completed a first abstract algebra course and four doctoral students working on algebraic topics were observed as they proved two group theory theorems and attempted to prove or disprove whether specific pairs of groups are isomorphic: $\mathbf{Z}_n$ and $\mathbf{S}_n$, $\mathbf{Q}$ and $\mathbf{Z}$, $\mathbf{Z}_p \times \mathbf{Z}_q$ and $\mathbf{Z}_{pq}$ (where p and q are coprime), $\mathbf{Z}_p \times \mathbf{Z}_q$ and $\mathbf{Z}_{pq}$ (where p and q are not coprime), $\mathbf{S}_4$ and $\mathbf{D}_{12}$. Nine times these undergraduates, who were successful in only two of twenty instances, first looked to see if the groups had the same cardinality; after which they attempted unsuccessfully to construct an isomorphism between the groups. They rarely considered properties preserved under isomorphism, despite knowing them (as ascertained by a subsequent paper-and-pencil test). For example, they all knew $\mathbf{Z}$ is cyclic, $\mathbf{Q}$ is not, and a cyclic group could not be isomorphic to a non-cyclic group, but they did not use these facts and none were able to show $\mathbf{Z}$ is not isomorphic to $\mathbf{Q}$, until afterwards. These facts did not seem to come to mind spontaneously, or in reaction to this kind of question.

In contrast, the doctoral students, who were successful in comparing all of the pairs of groups, rarely considered the definition of isomorphic groups. Instead, they examined properties preserved under isomorphism. When the groups were not isomorphic, they showed one group possessed a property that the other did not; for example, $\mathbf{Z}$ is cyclic, but $\mathbf{Q}$ is not. To prove $\mathbf{Z}_p \times \mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$, where p and q are coprime, three of them noted that the two groups have the same cardinality and showed $\mathbf{Z}_p \times \mathbf{Z}_q$ is cyclic. None tried to construct an isomorphism (Weber & Alcock, 2004).

Knowing which Theorems are Important

In comparing the proving behaviors of four undergraduates who had just completed abstract algebra and four doctoral students who were writing dissertations on algebraic topics, it was found that the doctoral students had knowledge of which theorems were important when considering homomorphisms. For example, in considering the proposition: *Let G and H be groups. G has order pq (where p and q are prime). f is a surjective homomorphism from G to H. Show that G is isomorphic to H or H is abelian,* all four doctoral students recalled the First Isomorphism Theorem within 90 seconds. In contrast, two undergraduates did not invoke the theorem, while the other two invoked its weaker form only after considerable struggle. When the doctoral students were asked why they used such sophisticated techniques, a typical response was, "Because this is such a fundamental and crucial fact that it's one of the first things you turn to" (Weber, 2001).

Another four undergraduates, who had recently completed their second course in abstract algebra, and four mathematics professors, who regularly used group-theoretic concepts in their research, were interviewed about isomorphism and proof (Weber & Alcock, 2004). They were asked for the ways they think about and represent groups, for the formal definition and intuitive descriptions of isomorphism, and about how to prove or disprove two groups are isomorphic. The algebraists thought about groups in terms of group multiplication tables and also in terms of generators and relations, as well as having representations that applied only to specific groups, such as matrix groups. Each algebraist gave two intuitive descriptions of groups being isomorphic: that they are essentially the same and that one group is simply a re-labeling of the other group. To prove or disprove two groups are isomorphic, they said they would do such things as "size up the groups" and "get a feel for the groups," but could not be more specific. In addition, they said that they would consider properties preserved by isomorphism and facts such as $\mathbf{Z}_n$ is *the* cyclic group of order n.

In contrast, none of the undergraduates could provide a single intuitive description of a group; for them, it was a structure that satisfies a list of axioms. While all four undergraduates could give the formal definition of isomorphic groups, none could provide an intuitive description. To prove or disprove that two groups were isomorphic, these undergraduates said they would first compare the order (i.e., the cardinality) of the two groups. If the groups were of the same order, they would look for bijective maps between them and check whether these maps were isomorphisms (Weber & Alcock, 2004).

It may be that undergraduates mainly study completed proofs and focus on their details, rather than noticing the importance of certain results and how they fit together. That is, they may not come to see some theorems as particularly important or useful. The mathematics education research literature contains few specific teaching suggestions on how to help students come to know which theorems are likely to be important in various situations. But, it might be helpful to discuss with them: (1) which theorems and properties you (the teacher) think are important and why, (2) your own intuitive, or informal ideas, regarding concepts, and (3) the advantages and disadvantages of various representations.

Teaching Proof and Proving

Some Suggestions Emanating from Research

One very positive finding, which was described earlier, is the remarkable sophistication of reasoning reached by some average school students who received brief interventions over a number of years (Maher & Martino, 1996a, 1996b, 1997). As described above, these students used a variety of spontaneously developed heuristics. Eventually, in order to come to agreement, these students, more or less, invented the idea of proof in a concrete case. If grade school students can be encouraged in this way, why not university students? Perhaps this could be done in part with relatively short "interventions" spread across the entire undergraduate program.

Another result is that younger students seem to prefer explanatory proofs written with a minimum of notation. This was certainly the case for U.K. Year 10 students (Healy & Hoyles, 1998). For example, instead of using mathematical induction to prove that sum of the first n integers is $n(n+1)/2$, one could use a variant of Gauss's original argument. Namely, for any n, one can write the sum in two ways as $(1 + 2 + 3 + \cdots + n)$ and as $(n + (n-1) + (n-2) + \cdots + 1)$, then add corresponding terms to obtain n identical summands equal to $n+1$, so twice the original sum equals $n(n+1)$. Hence, the original sum must equal $n(n+1)/2$ (Hanna, 1989, 1990). It seems plausible that undergraduates, and people more generally, might prefer proofs that provide insight to proofs that just establish the validity of a result.[20]

It also appears that great care should be taken to distinguish empirical reasoning from mathematical proof. Exactly how this can be done effectively is not especially clear, since merely giving high school geometry students a short instructional unit on this distinction left some of them very unclear as to the difference between empirical evidence and proof (Chazan, 1993). Perhaps secondary and university teachers need to stress this distinction often and also get students to discuss and reflect on situations where simple pattern generalization does not work.

Since current secondary teachers' conceptions of proof are somewhat limited and they sometimes accept non-proofs as proofs (Knuth, 2002a, 2002b), one way to enhance preservice secondary teachers' abilities to check the correctness of proofs might be to have them consider and discuss, in groups, a variety of student-generated "proofs," as well as having them provide feedback on each other's proofs.

In addition to explaining the difference between descriptive definitions in a dictionary and mathematical definitions, one can engage students in the defining process. For example, when using Henderson's (2001) investigational geometry text, one can begin with a definition of triangle initially useful in the Euclidean plane, on the sphere, and on the hyperbolic plane, but eventually students will notice that the usual Side-Angle-Side Theorem (SAS) is not true for all triangles on the sphere. At this point, they can be brought to see the need for, and participate in developing, a definition of "small triangle" for which SAS remains true on the sphere.

Perhaps it would also be possible to create classroom activities to improve students' ability to enhance their concept images and deal with representations flexibly. One suggestion is that upon introducing a new definition, one could ask students to generate their own examples, alternatively, to decide whether professor-provided instances are examples or non-examples, "without authoritative confirmation by an outside source" (Dahlberg & Housman, 1997, p. 298). Another possibility might be to engage students in conjecturing which kinds of symbolic representations might be useful for solving a given problem or proving a specific result. Also, one could point out that when a theorem has a negative conclusion (e.g., $\sqrt{2}$ is irrational), a proof by contradiction may be just about the only way to proceed.

For certain theorems in number theory, it has been suggested that the transition to formal proof can be aided by going through a (suitable) proof using a generic example

[20] It has been suggested that proofs have various functions within mathematics: explanation, communication, discovery of new results, justification of a new definition, developing intuition, and providing autonomy (e.g., Hanna, 1989; de Villiers, 1990; Weber, 2002).

that is neither too trivial nor too complicated (Rowland, 2002). Gauss's proof that the sum of the first n integers is $n(n+1)/2$, done for $n = 100$ is one such generic proof. Done with care, going over generic proofs interactively with students could enable them to "see" for themselves the general arguments embedded in the particular instances. If the theorem involves a property about primes, 13 and 19 are often suitable, provided the proof is constructive and that prime (e.g., 13) can be "tracked" through the stages of the argument. A generic proof, but not the standard one, can be given for Wilson's Theorem: *For all primes p*, $(p-1)! \equiv p-1 \pmod{p}$. That argument for p

be encouraged to write parts of a tentative proof "out of order" (e.g., What will the last line say?), even when they sometimes resist doing so.

There seems to be quite a lot to learn about the way in which proofs are customarily written. If students were taught about this way of writing in some of their courses, they might not be so puzzled about how to begin a proof. Indeed, we take the point of view that proofs are deductive arguments *in an identifiable genre*. They differ from arguments in legal, political, and philosophical works. Within this genre, individual styles can vary, just as novels by Hemingway and Faulkner have differing styles, although their novels are easily seen as belonging to a single genre that clearly differs from newspaper articles, short stories, or poems. As part of some ongoing work, we have been collecting general features of the genre of proof. For example, definitions already stated outside of proofs tend not to be written into them. In teaching, we have found that pointing out such features, especially in the context of a student's own work, can be helpful to students.

Furthermore, we have found it useful to have students carefully examine the structure of the statement that they are trying to prove, and even to think about how a tentative proof might be structured, before launching into it. For example, consider proving the theorem (mentioned earlier): *Let f and g be functions on A. If $f \circ g$ is one-to-one, then g is one-to-one.* It would be useful for a prover to first unpack the meaning of g being one-to-one. Doing so can direct one to begin the proof by writing, "Let x and y be in the domain of g and suppose $g(x) = g(y)$." This also makes clear that the desired conclusion is "Thus $x = y$." In this way, one exposes the "real, but hidden" mathematical task, namely, to get from $g(x) = g(y)$ to $x = y$. After that, students can concentrate on how the hypothesis that $f \circ g$ is one-to-one might help.

Concluding Remarks

We have tried to provide readers with a coherent organization of some of the mathematics education research on proof and proving, but there is much more.[22] Awareness of the variety of difficulties undergraduates have with proof and proving can make one more sensitive regarding how to help them. The above pedagogical suggestions indicate some steps one might take; however, more information on "what works" is needed.

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[22] Anyone wanting to delve into the considerable literature on proof and proving can go to the bibliography maintained by the *International Newsletter on the Teaching and Learning of Proof* at: http://www.lettredelapreuve.it/. Those with a more philosophical bent might want to consult the annotated bibliography at: http://fcis.oise.utoronto.ca/~ghanna/mainedu.html.

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