TENNESSEE TECHNOLOGICAL UNIVERSITY

1915

# Idempotents of Clifford Algebras

R. Abłamowicz *)
*Department of Mathematics, Tennessee Technological University*

B. Fauser ‡)
*Fachbereich Physik, Universität Konstanz*

K. Podlaski, ‖), J. Rembieliński**)
*Department of Theoretical Physics, University of Lódź*

Received 09-08-2003

A classification of idempotents in Cli ord algebras $C^{p,q}$

Table 1. Isomorphisms of $C^{p,q}$ with matrix rings [4].

| | | | |
|---|---|---|---|
| when | $p - q \pmod 8 = 3, 7$ | then | $C^{p,q} \approx M(2^{\frac{p+q-1}{2}}, \mathbb{C})$ |
| when | $p - q \pmod 8 = 2, 0$ | then | $C^{p,q} \approx M(2^{\frac{p+q}{2}}, \ )$ |
| when | $p - q \pmod 8 = 4, 6$ | then | $C^{p,q} \approx M(2^{\frac{p+q-2}{2}}, \ )$ |
| when | $p - q \pmod 8 = 1$ | then | $C^{p,q} \approx M(2^{\frac{p+q-1}{2}}, \ ) \quad M(2^{\frac{p+q-1}{2}}, \ )$ |
| when | $p - q \pmod 8 = 5$ | then | $C^{p,q} \approx M(2^{\frac{p+q-3}{2}}, \ ) \quad M(2^{\frac{p+q-3}{2}}, \ )$ |

## 2 $p - q$ ( 8) = 3, 7

Let us start with case $C^{p,q} \approx M(2^{\frac{p+q-1}{2}}, \mathbb{C})$. Suppose that $M(n, \mathbb{C})$ with $N = 2^{\frac{p+q-1}{2}}$ is an idempotent matrix. By means of a similarity transformation it can be transformed to its Jordan form. Consequently, the idempotency implies that the Jordan form of must be (up to a transposition of the basis vectors) of the form $\Pi_n = \mathrm{diag}\{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{0, \ldots, 0}_{N-n}\}$, $n = 0, 1, 2, \ldots, N$; note two trivial cases:

$n = 0$ ( = 0) and $n =$ =(t n(f)1.9(oha9.6( )1 1 3o)-35548.349.6( )]TJ /F12 5 0 TD 0 Tc (-13Tj 1 )Tj07F10 1 Tf 0.8

**3** $p - q$ ( 8) = 2, 0

In this case $C^{p,q} \approx M(2^{\frac{p+q}{2}}, \ )$. In complete analogy to the previous case we obtain the following classification of idempotents:

$$\mathrm{IDEM}_n(C^{p,q}) \approx \quad (N, \ )/( \quad (n, \ ) \times \quad (N-n, \ )),$$

where $N = 2^{\frac{p+q}{2}}$, $n = 0, 1, 2, \ldots, N$. In particular,

$$\dim_{\mathbb{R}} \mathrm{IDEM}_n = 2n(N-n).$$

**4** $p - q$ ( 8) = 4, 6

In this case $C^{p,q} \approx M(2^{\frac{p+q}{2}}, \ )$. As before[1]) we easily conclude that in this case

$$\mathrm{IDEM}_n(C^{p,q}) \approx \quad (N, \ )/( \quad (n, \ ) \times \quad (N-n, \ )).$$

$N = 2^{\frac{p+q-2}{2}}$, $n = 0, 2, 4, \ldots, N$, and[2])

$$\dim_{\mathbb{R}} \mathrm{IDEM}_n(C^{p,q}) = 8n(N-n).$$

**5** $p - q$ ( 8) = 1

In this case $C^{p,q} = M(2^{\frac{2+q-1}{2}}, \ ) \quad M(2^{\frac{2+q-1}{2}}, \ )$. Taking into account idempotency condition we can diagonalize idempotent in this case to the form $\Pi_{n,m} = \mathrm{diag}\{\underbrace{1, 1, \ldots, 1}_{n}, \underbrace{0, \ldots, 0}_{N-n}; \underbrace{1, 1, \ldots, 1}_{m}, \underbrace{0, \ldots, 0}_{N-m}\}$ where $N = 2^{\frac{p+q-1}{2}}$

$N = 2^{\frac{p+q-3}{2}}$, $n = 0, 2, 4, \ldots, N$, $m = 0, 2, 4, \ldots, N$, and

$$\dim_{\mathbb{R}} \mathrm{IDEM}_{n,m} = 8(n(N - n$$

then we find

$$= \frac{1}{2}(e^0 + {}_1e^1 + \hat{}_2e^2 + \hat{}_3e^3 - \hat{}_1e^{12} + {}_2e^{23} - {}_3e^{31}) .$$

In both cases, and satisfy (1) and (2).

## . T. . $p - q$ ( 8) = 2, 0

The lowest dimensional Cli ord algebras in this case are $C^{2,0}$ and $C^{1,1}$. We express all elements of these algebras in the following basis:

$$C^{2,0} = \text{span}\{I, \sigma^3, \sigma^1$$

We have demonstrated that idempotents of Cli ord algebras form algebraic varieties. These are smooth manifolds in the natural topology. Since idempotents can be used to generate left and right spinor ideals which carry representations of the algebra under consideration, these varieties contain valuable information about invariant subspaces and the representations. Since Cli ord algebras can in general be obtained by tensoring lower dimensional Cli ord algebras, present work provides information about tensor product representations that is valuable for branching processes and, possibly, for quantum information processing.

An obvious task for future research is to study intertwiners between the idempotent orbits. This is related to topological questions about the Witt ring on representations and should then be related to Brauer-Wall groups. One might expect to gain insight via, for example, Hopf fibration of spheres into the number and mutual relations of the intertwiners. Furthermore, it may be promising to reinspect the Radon-Hurwitz number involved in the matrix algebra isomorphisms, which counts the number of global vector fields, by performing the present work directly in an algebraic setting without using the matrix representations.

## References


[1] C. Chevalley: *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[2] P. Lounesto, *Cli ord Algebras and Spinors*, 2nd ed, Cambridge University Press, Cambridge, May 2001.

[3] R. Ablamowicz: *Spinor representations of Cli ord algebras: A symbolic approach*, Comp. Phys. Comm. Thematic Issue Computer Algebra in Physics Research Vol. 115, December 11, 1998, Numbers 2–3, pages 510–535.

[4] J. Rembieliński: in *Cli ord Algebras and their Applications in Mathematical Physics*


6(ich)-eBos T8 (19oiv)n.2(,95(A63TdsT8A[.4M(d8i9e12af6,9)3K.26ce)-IA7(ae)-19.4(-3],70142372-0,3pD 0.D835(lJ7-1t)]Tn2i-2;369770r -1.64(p)694.6729 TD -0.0049 Tc