DEPARTMENT OF MATHEMATICS
TECHNICAL REPORT

ERRORS AND MISCONCEPTIONS IN COLLEGE LEVEL THEOREM PROVING

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AUGUST 2003

No. 2003-3

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This paper first appeared as

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INTRODUCTION

In this paper we describe a number of types of errors and underlying misconceptions that arise in mathematical reasoning. Other types of mathematical reasoning errors, not associated with specific misconceptions, are also discussed. We hope the characterization and cataloging of common reasoning errors will be useful in studying the teaching of reasoning in mathematics.

Reasoning in mathematics is no different from reasoning in any other subject. Mathematics does, however, contain many exceedingly long and intricate arguments. The understanding of such arguments is essential to the correct application of mathematics as well as to its continued development. Indeed, in mathematics it is the proofs that provide the strong consensus on the validity of basic information that is characteristic of a science. From this point of view, the idea of proof in mathematics corresponds to those special techniques of observation and experimentation which are essential to the other sciences.

The examples presented here arose as student errors in a junior level course in abstract

Of course this simple view of the nature of lower division college mathematics leads to a simple view of how it should be taught. The concepts should be explained, the algorithms should be provided, and the implementation of the algorithms should be practiced.

We think this is the conception of mathematics and its teaching held by many college students. We do not claim that they consciously ascribe to a description such as this or that they think about this topic at all. Rather, we note that they act as if they did and that they do so persistently.

Any teacher of lower division college mathematics can observe this by asking students to solve problems for which they have not been provided algorithms. Even if the teacher explains

Clearly, if mathematics is to be widely useful, the static view of it is inadequate. Even on the lower college level, mathematics should include the creation of algorithms, at least in the sense of combining and altering known techniques. Since correctness of a newly created algorithm cannot be ascertained by appeals to authority, reasoning should also be regarded as an integral part of mathematics and its teaching. We do not mean to suggest that reasoning on this level should be in the form of proofs, but some examination of the correctness of algorithms should be included.

Perhaps this point can best be illustrated by looking outside mathematics to programming courses. Students are required to produce their own algorithms and it often happens that these algorithms do not perform as expected. As a result, the validation of programs through testing is regarded as an important and necessary component of the discipline.

In the mathematical community this technique is sometimes called the Moore method after the late R. L. Moore who practiced it with remarkable success (Forbes, 1971). Of course in a broader setting this sort of teaching has a long history and calls to mind the methods of Socrates. There are many versions of this method, and seemingly insignificant variations in it may greatly alter its effectiveness.

REASONING ERRORS

Students make a great variety of reasoning errors in attempting proofs. We feel some of these errors are based on underlying misconceptions, while others, although repeatedly observed, are of a technical or other nature. Students persist in making both types of errors[2]. For the former, we offer our views as to the possible underlying misconceptions, that is, we give a general rule or idea which, if believed by a student, would result in that type of error. These errors are taken to have a rational basis, and we comment on how they might come about[3]. For the latter, we sometimes speculate on the underlying causes of the errors, but do not see them as conceptual in nature. Each type of error is illustrated with one or more actual student "proofs" [4].

REASONING ERRORS BASED ON MISCONCEPTIONS

M1. Beginning with the conclusion, arriving at an obvious truth and thinking the proof is complete. Of course, this provides a valid argument if and only if the steps are reversible.

The misconception consists in thinking that one valid technique of proof begins with the conclusion and ends with a known fact. However, this is not acceptable as it is often difficult to arrange a proof into a sequence of discrete steps, each of which can easily be checked for reversibility. This misconception may have arisen from methods learned in secondary school for verifying trigonometric identities and solving equations. Also, since a good heuristic for discovering a proof is to analyze the meaning of the conclusion, college students may have seen this presented in class, along with a statement that all steps are reversible. They, thus, could easily be confusing discovery with proof.

Example

Theorem: Let G be a group such that for all g in G, $g^2=e$, where e is the identity of the group. Then,

(i) for all g in G, $g = g^{-1}$ and,
(ii) G is commutative.

"Proof" of (ii) having proved (i): To show G is commutative means, for all a and b in G, it must be that $ab=ba$. Multiplying $ab=ba$, by the appropriate inverses, and using part (i), one gets $a=bab$, and $b=aba$. Now,

(1) $b * = aba * = (bab)ba=ba(bb)a=baea=baa=be=b$,

and

(2) $a * = bab * = (aba)ab=ab(aa)b=abeb=abb=ae=a$.

Comment

There are no errors present in lines (1) and (2). In this case, the steps indicated with a "*" are not reversible; they are equivalent to $ab=ba$.

M2. Names confer existence. This error occurs from failing to distinguish between symbols for things whose existence is established and symbols for things whose existence is not established. Often this error occurs when a student attempts to solve an equation, without questioning whether a solution exists. We asked a class of precalculus college students to solve the equation $\cos x = 3$ in order to test whether they could apply the fact that the range of $\cos x$ is [-1,1]. Most tried unsuccessfully to manipulate the equation to find x and were unable to reach the proper conclusion.

A different example of this type of error occurred when an abstract algebra student attempted to prove that a semigroup in which the equations $ax=b$ and $ya=b$ always have solutions is a group. One must first establish the existence of an identity element, e, and then show that each element, g, in the group has an inverse. The student attempted the second part by contradiction, supposing that g had no inverse. Then, in the very next line, he used the symbol g^{-1}, which he just assumed didn't exist, and made calculations with it.

The underlying misconception is that names always represent existing things. Writing $\cos x$ or g^{-1} seems to confer existence on and the right to manipulate the symbol. Perhaps this comes from secondary school algebra where x is referred to as “the unknown”, that is, as something which exists and should be found.

If one asks an abstract algebra student to prove that the equation $ax=b$ has a solution in a group, he will often proceed as follows: $ax=b$, so $a^{-1}ax=a^{-1}b$, so $x=a^{-1}b$. He does not realize that by manipulating the equation he is tacitly assuming x exists. Of course, he should produce a group element, in this case $a^{-1}b$, which when substituted in the given equation yields a true statement.

One final simple-minded example illustrates the difficulties that can occur. Given $(x+3)(x+2) = (x+1)(x+4)$, one can conclude $6 = 4$, by supposing there is such an x.

M3. Apparent differences are real. This error occurs when things which have different names are taken to be different. The underlying misconception is there is a one to one correspondence between names and mathematical objects.

Although students realize that the same real number can be written in many different ways, for example, $1/2 = 2/4$ or $3 = 1+2$, often it does not occur to them that two different abstract expressions may represent the same thing. This happens even though they know two apparently different trigonometric expressions can be equal from having verified" identities.

Example

<u>Theorem:</u> If a commutative group has an element of order 2 and an element of order 3 then it must have an element of order 6.

"Proof": Let g be the element of order 3 and let h be the element of order 2. The $g^3 = e$ and $h^2 = e$ where e is the identity of the group.

Consider the subgroup generated by hg. Since $h^6g^6 = (h^2)^3 (g^3)^2 = e^3e^2 = e$, this subgroup is $\{hg, h^2g^2, h^3g^3, h^4g^4, h^5g^5, h^6g^6\}$ which simplifies to $\{hg, g^2, h, g, hg^2, e\}$ using $g^3=e$ and $h^2=e$. So hg has order 6.

Comment

Something is missing here, namely an argument showing that the 6 symbols hg, g^2, h, g, hg^2, e represent 6 distinct elements. This can be shown, but one shouldn't assume the student can show it or is even aware that he must.

Example

Lagrange's Theorem: Let G be a group of order n. Let H be a subgroup of order m. Let r be the number of distinct right cosets of H in G. Then $n=rm$.

Comment

Example

<u>Theorem:</u> A group G in which every element is of order 2 is commutative.

"Proof": Let *g*, *h* be elements of G. By hypothesis,

OTHER ERRORS

El. Overextended symbols. This error occurs when one symbol is used for two distinct things, often because the distinction was unobserved. Such errors can indicate an incomplete grasp of a mathematical structure, such as group, and first appear when the structure is used several times in the same setting.

Example

<u>Theorem:</u> Let G_1 and G_2 be two groups contained in a semigroup S such that $G_1 \cap G_2$ is

“Proof”: Let G be a commutative group, H$\subseteq$G. Let g_1 *e* G, g_2 *e* H, g_3 *e* G. According to Theorem 55 of the notes (on using a subgroup H to define the right coset equivalence relation on G), H $g_1 \subseteq$ G. Then H $g_l = g'$ *e* G. Let *a* be the order of g', which is the number of distinct right cosets of H in G. Let *b* be the order of $g_{2,}$ *c* be the order of g_3. Since g_3 *e* G, *c* is the order of G. We want to show $c=ab$. According to Lagrange's Theorem, the order of a group is equal to the order of a subgroup times the number of right cosets of that subgroup in the group. Therefore, $c=ab$. In this theorem $a=2$, $b=3$, $c=2\cdot 3=6$. This gives an element of order 6.

Comment

On the surface this appears to be a proof. It starts and stops in an expectable way and quotes Lagrange's Theorem correctly. It is also syntactically correct, except for one small place. However, it is impossible to find any basic underlying idea which the student might have started with or to follow the individual sentences[7].

The student may have selected Lagrange's Theorem almost arbitrarily[8] and tried to develop it into a proof.

E7. Substituting with abandon. This error consists in obtaining one statement from another using an unjustifiable substitution. One fixed element is replaced by another unequal fixed element. Of course, it is permissible to substitute for a universally quantified variable; and perhaps, this error results from confusing the two situations.

This error often occurs when a student attempts to prove a theorem which begins “For all *s* in S, …”. The standard way to start the proof is to write “Let *s* *e* S.” With these words, *s* becomes a fixed element, and one is no longer free to substitute for *s*. Occasionally, to emphasize this point, one writes, "Let s be an arbitrary, but fixed element of S."

Examples

Theorem (Cancellation Law): Let G be a group. Let *g*, *h*, *k* be elements of G. If $gh=gk$, the $h=k$.

“Proof”: Let *e* be the identity of G. Substitute *e* for *g* in $gh=gk$. The $eh=ek$, so $h=k$.

Theorem: Let G be a group with identity *e*. If *g*, *h*, *k* are elements of G so that $gh=e=hg$ $h=k$

“Proof”: Theorem S

Example

Theorem: If G is a group of order n, then $g^n = e$ for all g in G, where e is the identity of G.

"Proof": Since G is finite, and one wants to show $g^n = e$ for all g in G, one can choose n to be zero or a value which gives the identity element.

Comment

The student has failed to realize n

E10. Using information out of context. In this type of error information from one argument is improperly used in another, often because identical symbols appear in both. This error is most likely to occur when proofs are organized into independent sections, for example, in theorems involving case analysis, set equality, or equivalence of statements. In such situations, a student may unjustifiably transfer information from one section to another. The next example is rather unusual in that the error involves two theorems, rather than two independent sections of one proof.

Example

Theorem (Cancellation Law: Let G be a group. Let *g*, *h*, *k* 1.

2. How does each of these reasoning errors arise and how could it be prevented?
3. Is the making of one type of reasoning error correlated with making others? Perhaps students who make a particular type of error always make another type.
4. Which types of reasoning error occur most frequently?
5. Do certain types of reasoning errors occur more in one course than another, for example, in algebra as compared with topology?
6. Are any of these reasoning errors correlated with particular sections of students' earlier coursework?

If lower division mathematics courses were to ignore the static view and include significant instruction on creating and validating algorithms, it is possible that reasoning would be improved, as well as applications extended. Evidence concerning this point would be useful.

Finally, we note that there is remarkably little correlation between the reasoning errors we have observed and classified and the topics emphasized in an introductory logic course or even in one of the newer courses as on transitions to advanced mathematics[9].

<u>END NOTES</u>

[1] A detailed description of this method is given in Selden and Selden (1978, p. 69-71).

[2] This agrees with the general research on misconceptions (Novak, 1983, p.2).

[3] An assumption shared by Confrey. (1983, p.30, p.25).

[4] Occasionally extraneous material was edited out.

[5] Lack of precision and accuracy have been observed in first year university students' attempts to solve physics problems (Mehl and Volmink, 1983, p.228). Lin (1983, p.202) has suggested that beginning physics students are unaccustomed to the necessary precision of expression.

[6] Students indicate they solve mathematics problems by imitating textbook examples (Confrey, 1983, p.23).

[7] The authors have made a painstaking attempt to find some basic underlying idea, plus a line- by-line analysis of this “proof” (Selden and Selden, 1978, p.78-9).

[8] In making this “proof”, the student appears to have acted in an unsystematic and somewhat impulsive way (Mehl and Volmink, 1983, p.228). He does not take what computer scientists would call a top-down approach.

[9] We recently taught such a course several times and found the treatment in a typical text (Smith, Eggen, and St. Andre, 1986) only superficially related to these reasoning errors.

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