DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

THE DENSE PACKING OF 13 CONGRUENT CIRCLES IN A CIRCLE

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Abstract. The densest packings of n congruent circles in a circle are known for $n \leq 12$ and $n = 19$. In this paper we exhibit the densest packings of 13 congruent circles in a circle. We show that the optimal con gurations are identical to Kravitz's [11] conjecture. We use a technique developed from a method of Bateman and Erdős [1] which proved fruitful in investigating the cases $n = 12$ and 19 by the author [6, 7] .

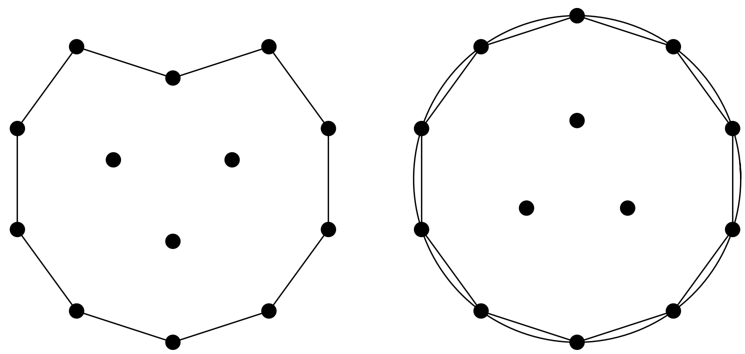
1. Preliminaries and Results

We shall denote the points of the Euclidean plane $\mathbf{E}^2$ by capitals, sets of points by script capitals, and the distance of two points by $d(P,Q)$. We use PQ for the line through P, Q, and $\overline{PQ}$ for the segment with endpoints P, Q. $\angle POQ$ denotes the angle determined by the three points P,O,Q in this order. (r) means the closed disc of radius r with center O. By an annulus $r < \rho \leq s$ we mean all points P such that $r < d(P,O) \leq s$. We utilize the linear structure of $\mathbf{E}^2$ by identifying each point P with the vector $\vec{OP}$, where O is the origin. For a point P and a vector $\vec{a}$ by $P + \vec{a}$ we always mean the vector $\vec{OP} + \vec{a}$.

The problem of finding the densest packing of congruent circles in a circle arose in the 1960s. The question was to find the smallest circle in which we can pack n congruent unit circles, or equivalently, the smallest circle in which we can place n points with mutual distances at least 1.

In this article we shall find the optimal configurations for $n = 13$. We are going to prove the following theorem.

Theorem 1. *The smallest ci cle $\mathcal{C}$ in which we can pack* 13 *points with mutual distances at least* 1 *has adius* $R = (2\sin 36°)^{-1} = \frac{1+\sqrt{5}}{2}$*. The* 13 *points fo m the following two configu ation as shown on Figu e 1.*

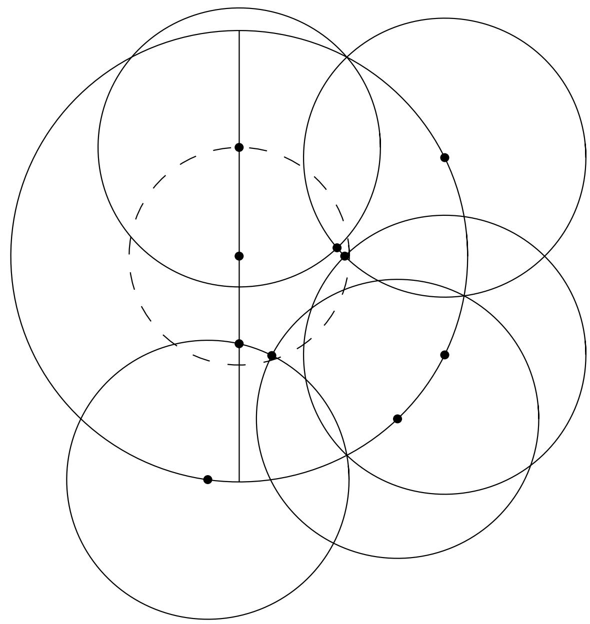


the 6 points either form a regular hexagon of unit side length with vertices on (1), or a pentagon with all 5 vertices on (1) and a sixth point at O. In both cases it is clear that there cannot be 7 points in the annulus $1 < \rho \leq R$. Note that if there are exactly 9 points in the annulus $1 < \rho \leq R$, then the 13 points must form the first configuration shown on Figure 1.

Lemma 3. *The e cannot be exactly 5 points in* (1).

P oof. Suppose, on the contrary, that there are 5 points in (1). Bateman and Erdős [1] proved that the radius of the circumcircle of 5 points with mutual distances at least 1 is $d_5 = (2\cos 54°)^{-1} = 0.85\ldots$. The minimal radius is realized by a regular pentagon of unit side length. According to Lemma 2 there must be two points P and

Notice that $\frac{r^2-s^2+1}{R-r^2}$ takes on its maximum if $r = s = 0.77$, and the maximum is less than 1. On the left hand side we can see that we decrease $(2rs)^2 - (s^2 + r^2\{$



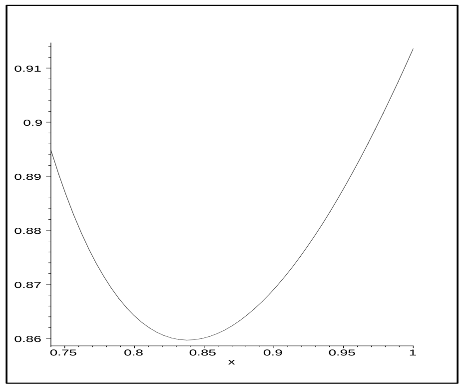


FIGURE 3

it may be written as a polynomial equation $p(Q_0, Q_6) = 0$ for d_1. The Cartesian coordinates of Q_0 and Q_6 are the following.

$Q_6 = (R(\sin(36° + \psi) - \sin\psi), d + R(\cos(36° + \psi) - \cos\psi))$;
$Q_0 = (0, -R\cos(36° - \psi) + \sqrt{R^2\cos^2(36° - \psi) - R})$
The polynomial equation is as follows.

$p(Q_0, Q_6) = (-10 - 4\sqrt{5})d_1^4 + (-5 - 7\sqrt{5})d_1^6 + 14\sqrt{5}d_1^8 + (55 + 13\sqrt{5})d_1^{10} + (25 + 15\sqrt{5})d_1^{12} + (10 + 4\sqrt{5})d_1^{14} = 0$

This equation has two roots in the $[\frac{\sqrt{2}}{2}, 1]$ interval, $0.744\ldots$ and 1. By direct substitution we can check that $p(Q_0, Q_6) < 1$ in $(0.7448, 1)$. In a similar manner we may write $d^2(Q_6, Q_7) = 1$ as a polynomial equation for d_1 and check for roots in the designated interval. Note that $d(O, Q_7) = R\cos(36° - \psi) - \sqrt{(1 - R^2\sin^2(36° - \psi))}$. The graph of the function $d(Q_6, Q_7)$ is shown on Figure 2.

This function has no zeros in the interval $[0.745, 1]$. □

Lemma 6. *If $P_7, P_{10}, P_{11} \in P_3OP_4$, then is not possible that $d_1 \in [\sqrt{2}/2, 0.745]$.*

P oof. For every value of d_1 there is a d_m such that none of the three points P_2, P_3, P_4 can be closer to O than d_m. We may obtain d_m from the the follwing equation.

s a r c $\phi(d_1, d_1) + \phi(d_1, d$ @P

is not possible that both d_3 and d_4 are less than or equal to d_M. This

Q_6. Therefore the four points in (1) must be in the configuration shown in the second part of Figure 1.

References

[1] P. Bateman, P. Erdős, *Geometrica extrema suggeste*