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# ON THE INVERTIBILITY OF SOME OPERATORS ON HILBERT SPACES

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For any given bounded linear operator $A$ on a complex Hilbert space $H$, we give su ! cient conditions to ensure the existence of a bounded operator $B$ on $H$ such that (i)$AB + BA$ is of rank one, and (ii)$I + e^{xP(A)+tQ(A)}B$ is invertible for all $x, t \in \mathbb{R}$ where $P(A)$ and $Q(A)$ are polynomials in $A$. Our main results will provide a justiÞca-tion in general terms to a crucial step of the so-called operator method used by Aden, Carl, and Schiebold [",3] to solve nonlinear partial di#erential equations like the Korteveg-deVries(KdV), modiÞed KdV, Kadomtsev-Petviashvili equations.

## 1. INTRODUCTION

In ["] Aden and Carl showed that for a given bounded linear operator $A$ on a Banach space $E$ the family of operators $V(x,t) := (I + L)^{-1}e^{Ax+A^3t}(AB + BA)$ is a solution to the operator KdV equation $V_t = V_{xxx} + 3V_x^2$, provided the operator $B$ satisÞes (i) $AB + BA$ is of rank one, and (ii) $(I + L)^{-1}$ exists, where $L(x,t) := e^{Ax+A^3t}B$. Further, $v(x,t) := tr(V(x,t)))$, where $tr$ is the continuous trace, gives a scalar solution to the scalar KdV equation $v_t = v_{xxx} + 3v_x^2$. A similar app… partial equations … nonlinear physics like the … other pa

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equation, Kadomtsev-Petviashvili equation as well as the sine-Gordon equation. The approach mentioned above is known as the operator method. The main idea of the operator method can be described as follows. Given a nonlinear PDE of soliton physics as well as a speciÞc scalar solution to the equation, the Þrst step in the solution is to translate the given nonlinear equation to an operator equation. Using the speciÞc scalar solution as an aid, one then searches for a family of operator solutions to the operator equation. Having obtained the operator solutions, the second step is to transfer the operator-valued solution

## 2. PRELIMINARIES

Recall that an operator $T : E \to F$ (where $E$ and $F$ are Banach spaces) is said to be of rank one if the dimension of the range of $T$ is equal to one. It is straightforward to verify that $T$ is of rank one if and only if there exists $a \in E'$ (dual of $E$) and $y \in F$ such that $T = a \otimes y$, where $(a \otimes y)x := a(x)y, \ \forall x \in E$. It is obvious that for any $a, b \in E'$; $x, y \in E$, and complex number , we have (i) $(a \otimes x) = a \otimes \ x$, (ii) $(a \pm b) \otimes x = a \otimes x \pm b \otimes x$, and (iii) $(a \otimes x) \circ (b \otimes y) = b \otimes a(y)x$.

The following lemmas are quite useful in the proofs of the main results of the paper. Lemma 2.

$x^t$ always denotes the corresponding column vector. For any square matrix A, $E(A)$ denotes the set of eigenvalues of A. Finally, I always stands for the identity operator.

## 3. FINITE DIMENSIONAL CASE

Recall that for any $h, g \in C^n$, $h \otimes g$ gives a linear operator on $C^n$ which is deÞned as follows $(h \otimes g)x = \sum_{i=1}^{n} \overline{h}_i x_i \; g$ for each $x \in C^n$. Even though

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for any $h, g \in C^n$ there always exists a matrix B such that $AB + BA = h \otimes g$ provided $0 \notin E(A) + E(A)$ for a given A (see ["], [4], [6]), we show that a careful choice of h and g will also ensure the invertibility of the matrix $I + e^{xP(A)+tQ(A)}B$

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exists a diagonal matrix $\tilde{A}$ such that $\tilde{A} = SAS^{-1}$. Let $\{e_j\}$ be the standard basis, and let $\tilde{A}e_j = \mu_j e_j$, " $\leq j \leq n$. DeÞne the following matrix $\tilde{B} := (b_{ij})_{i,j=1}^{n}$ where $b_{ij} = \frac{{}_i \bar{{}_j}}{\mu_i + \mu_j}$, $= ( {}_1, \ldots, {}_n)^t$, and $= ( {}_1, \ldots, {}_n)^t$. Note that the entries $(b_{ij})$ of the matrix $\tilde{B}$ are well-deÞned since $0 \notin E(A) + E(A)$.

Claims: (") $\tilde{A}\tilde{B} + \tilde{B}\tilde{A} = \ \otimes\ $.

(2) The following choice of and guarantees the invertibility for the matrix $(I + e^{xP(\tilde{A})+tQ(\tilde{A})}\tilde{B})$ for all $x, t \in R$: for $i < n$, ${}_i = 0$ if $i$ is odd, ${}_i \neq 0$ if $i$ is even; ${}_i \neq 0$ if $i$ is odd, ${}_i = 0$ if $i$ is even, and $\frac{{}_n \bar{{}_n}}{2\mu_n} \geq 0$.

Proof of claim ("). It is enough to show that for each $j$, " $\leq j \leq n$

$(\tilde{A}\tilde{B} + \tilde{B}\tilde{A})e_j = ( \ \otimes\ )e_{jj}$.j ≤ nB We have $(\tilde{A}\tilde{B} + \tilde{B}\tilde{A}$

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Now we are ready to exhibit $h, g \in C^n$ and an $n \times n$ matrix $B$ as stated in the theorem. Let $B := S^{-1}\tilde{B}S$,

$h := \ _2(\bar{s}_{21}, \ldots, \bar{s}_{2n})^t + \ _4(\bar{s}_{41}, \ldots, \bar{s}_{4n})^t + \ldots + \ _n(\bar{s}_{n1}, \ldots, \bar{s}_{nn})^t$ and

$g := \ _1(q_{11}, \ldots, q_{n1})^t + \ _3(q_{13}, \ldots, q_{n3})^t + \ldots + \ _n(q_{1n}, \ldots, q_{nn})^t$

where $S = (s_{ij})$ and $S^{-1} = (q_{ij})$. Notice also that $S^{-1}( \ \otimes \ )S = h \otimes g$. Then

$AB + BA = S^{-1}\tilde{A}SS^{-1}\tilde{B}S + S^{-1}\tilde{B}SS^{-1}\tilde{A}S = S^{-1}(\tilde{A}\tilde{B} + \tilde{B}\tilde{A})S = h \otimes g.$

This proves part (i) of the theorem. Since

$$I + e^{xP(A)+tQ(A)}B = S(I + e^{xP(\tilde{A})+tQ(\tilde{A})}\tilde{B})S^{-1}$$

and $I + e^{xP(\tilde{A})+tQ(\tilde{A})}\tilde{B}$ is invertible, statement (ii) of the theorem follows. □

Now we are ready to extend the above theorem to a general situation.

Theorem 3.2. Let $A$ be any square matrix of size $n$ such that $0 \notin E(A) + E(A)$. Then there always exist non-zero vectors $h, g \in C^n$ and an $n \times n$ matrix $B$ such that

(i) $AB + BA = h \otimes g$, and

(ii) $I + e^{xP(A)+tQ(A)}B$ is invertible for all $x, t \in R$ where $P(A)$ and $Q(A)$ are polynomials in $A$.

Proof. Let $S$ be such a similarity transformation which reduces the matrix $A$ to a Jordan canonical form in which all the " × " Jordan blocks appear at the bottom of the matrix $\tilde{A}$. Obviously, this can always be achieved since the Jordan canonical form is unique up to permutations of the Jordan blocks. The matrix $\tilde{A}$ will be as follows

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we extend the above result to a more general situation by using the results of the previous section.

Suppose

$$A = \begin{pmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & A_3 & & & \\ & & & \ddots & & \\ & & & & A_k & \\ & & & & & \ddots \end{pmatrix}$$

is the matrix representation of an operator on $\ell_2$ in the standard basis $\{e_j\}$, where $A_k$ $(k \in \mathbb{N})$ is a normal square matrix of size $n_k$ placed on the main diagonal of the infinite matrix $A$ and there exists a positive integer $n_0$ such that the size of each $A_k$ is less than or equal to $n_0$. All the entries of $A$ which are not shown are equal to zero. Assume that there exists a real number $M > 0$ such that $\|A_k\| \leq M$ for all $k$. Here $\|A_k\|$ is the Euclidian norm of the matrix $A_k$. Also assume that all eigenvalues $\{\mu_i\}$ of $A$

Proof. Obviously, the operators $A, D, U$ and $U^{-1}$ are linear. Let $x \in \ell_2$. For convenience we write $x$ in cycles $x = (x_1^{(1)}, \ldots, x_{n_1}^{(1)}; \ldots, x_1^{(k)}, \ldots, x_{n_k}^{(k)}; \ldots)$, where $n_k$ is the size of the matrix $A_k$, $k = 1, 2, \ldots$ . To show that $A$ is bounded note that $Ax = (A_1 r_1, \ldots, A_k r_k, \ldots)$ where $r_k = (x_1^{(k)}, \ldots, x_{n_k}^{(k)})$.

Then $\|Ax\|^2 = \sum_{k=1}^{\infty} \|A_k r_k\|^2$. Further, by using Hölder's inequality

$\|A_k r_k\|^2 = |a_{11}^{(k)} x_1^{(k)}$ I21# $+ \ldots + a$

the domain of $U^{-1}$ (which is the range of U) is closed [8]. Hence, range(U)=$\grave{}_2$.

Before proceeding further, we need two lemmas. The operators R and S deÞned in the following lemmas are used in [3]. Even though some of the properties of R and S are implicitly present in [3], we provide some explicit proofs for the sake of completion.

Lemma 4.3. Let $= \{ {}_i \} \in \grave{}_2$, $= \{ {}_i \} \in \grave{}_2$, and let $\{\mu_i\}$ be an infinite bounded sequence of positive real numbers such that ² = $\inf_{i \in N} \mu_i > 0$. Then

(i) $R : \grave{}_2 \to L_2(0, \infty)$ defined by $R(x_1, x_2, \ldots)(s) = {}_{j=1}^{\infty} \bar{}_j e^{-s\mu_j} x_j$ is a bounded linear operator, and

(ii) $S : L_2(0, \infty) \to \grave{}_2$ defined by $S(f) = (\ldots, {}_0^{\infty} f(s) e^{-s\mu_i} \; {}_i ds, \ldots)$ is a bounded linear operator.

Proof. Proof of (i). Clearly, R is a linear operator. Let $x = (x_i) \in \grave{}_2$. Then

$$ {}_0^{\infty} |\bar{}_j e^{-s\mu_j} x_j|^2 ds \; {}^{1/2} = |\bar{}_j x_j|^2 \; {}_0^{\infty} e^{-2s\mu_j} ds \; {}^{1/2} = \frac{|\bar{}_j x_j|}{\overline{2\mu_j}} \leq \frac{|\bar{}_j x_j|}{\sqrt{2}}. $$

Thus

$$ \sum_{j=1}^{\infty} \| \bar{}_j e^{-s\mu_j} x_j \|_{L_2} \leq \sum_{j=1}^{\infty} \frac{|\bar{}_j x_j|}{\sqrt{2}} < \infty. $$

Since $L_2(0, \infty)$ is a Banach space and absolute convergence in a Banach space implies convergence, it follows that ${}_{j=1}^{\infty} \bar{}_j e^{-s\mu_j} x_j$ is convergent in $L_2(0, \infty)$, and hence R is well-deÞned. To show that R is bounded, notice that by using the Hölder's inequality

$$kR(x_1, x_2, \ldots)k_{L_2} = \sum_{j=1}^{\infty} {}^{-}{}_j e^{-s\mu_j} x_j {}_{L_2} \leq \sum_{j=1}^{\infty} |{}^{-}{}_j x_j| \cdot ke^{-s\mu_j} k_{L_2}$$

$$\leq \sum_{j=1}^{\infty} \frac{|{}^{-}{}_j x_j|}{\sqrt{2}} \leq \frac{"}{\sqrt{2}} \sum_{j=1}^{\infty} |\ {}_j|^2$$

Theorem 4.5. Let $\{A_k\}$ be the sequence of normal matrices of bounded size such that all eigenvalues $\{\mu_i\}$ of $A_k$'s are positive and constitute a bounded sequence such that $\epsilon = \inf_{i \in N} \mu_i > 0$. Let $A : \ell_2 \to \ell_2$ be an operator constructed as in the discussion preceding the proposition 4.1. If there is a positive real number $M$ such that $\|A_k\| \leq M$ for all $k$, then there exist non-trivial vectors $h, g \in \ell_2$ and g

Thus

$$\langle (I + K_{x,t})f, f\rangle = \langle f, f\rangle + \langle K_{x,t}f, f\rangle = \|f\|^2 + \langle K_{x,t}f, f\rangle \geq 0$$

Hence, $-$"

be such that $c_k := \bar{}_k \; _k > 0$ for any $k \in N$. According to Lemma 2.2 the operator equation $D\tilde{B} + \tilde{B}D = \; \otimes \;$ has the solution $\tilde{B} = \int_0^\infty e^{-sD}( \; \otimes \; )e^{-sD}ds$. Thus, we have

$$\begin{aligned}
\tilde{B}(x_1,\ldots,x_n,\ldots) &= \int_0^\infty e^{-sD}( \; \otimes \; )(e^{-s\mu_1}x_1,\ldots,e^{-s\mu_n}x_n,\ldots)ds \\
&= \int_0^\infty \sum_{j=1}^\infty \bar{}_j e^{-s\mu_j} x_j (e^{-s\mu_1} \; _1,\ldots,e^{-s\mu_n} \; _n,\ldots)ds \\
&= (\ldots,\underbrace{\int_0^\infty \sum_{j=1}^\infty \bar{}_j e^{-s\mu_j} x_j e^{-s\mu_i} \; _i ds}_{\text{ith component}},\ldots) \\
&= SR(x_1,\ldots,x_n,\ldots)
\end{aligned}$$

where operators $R : \ell_2 \to L_2$ and $S : L_2 \to \ell_2$ are deÞned in Lemma 4.3. Then according to Lemma 2.",

$I + e^{xP(D)+tQ(D)}\tilde{B} = I + e^{xP(D)+tQ(D)}SR$ is invertible i# $I + Re^{xP(D)+tQ(D)}S$ is invertible. Now

$$Re^{xP(D)+tQ(D)}S(f)(w) = Re^{xP(D)+tQ(D)}(\ldots, \int_0^{\infty}$$

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$g := {}_1(q_{11}, \ldots, q_{n1}, \ldots)^t + \ldots + {}_n(q_{1n}, \ldots, q_{nn}, \ldots)^t + \ldots$

where $U = (u_{ij})$ and $U^{-1} = (q_{ij})$. Notice also that $U^{-1}(\quad \otimes \quad)U = h \otimes g$. Then one checks

$$AB + BA = U^{-1}DUU^{-1}\tilde{B}U + U^{-1}\tilde{B}UU^{-1}DU$$

$$= U^{-1}(D\tilde{B} + \tilde{B}D)U$$

$$= U^{-1}(\quad \otimes \quad)U = h \otimes g.$$

This proves part (i) of the theorem. Since

$$I + e^{xP(A)+tQ(A)}B = U(I + e^{xP(D)+tQ(D)}\tilde{B})U^{-1}$$

and $I + e^{xP(D)+tQ(D)}\tilde{B}$ is invertible, statement (ii) of the theorem follows. □

The previous theorem was a special case of the operator A made up of infinitely many normal matrices. What happens if all the conditions of Theorem 4.5 are satisfied except for the normality of matrices $A_k$? Thus, suppose an operator $A : \ell_2 \to \ell_2$ is given which has the following matrix representation in the standard basis $\{e_i\}$

$$A = \begin{pmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & A_3 & & & \\ & & & \ddots & & \\ & & & & A_k & \\ & & & & & \ddots \end{pmatrix}$$

where $A_k$ $(k \in N)$ is a square matrix of size $n_k$ placed on the main diagonal of the infinite matrix A, and there exists a positive integer $n_0$ such that the size of each $A_k$ is less than or equal to $n_0$. All the entries of A which are not shown are equal to zero. Assume that there exists a real number $M > 0$ such that $kA_k k \leq M$ for all k. Also assume that all eigenvalues $\{\mu_i\}$ of A are positive and