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ON OPTIMAL ROW-COLUMN DESIGNS FOR TWO TREATMENTS

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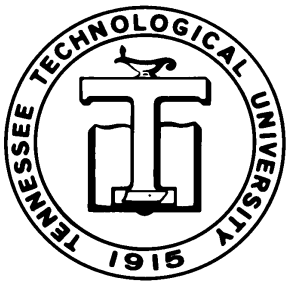

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On optimal row-column designs for two treatments

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Abstract: This paper presents optimal 3 3, 3 4, and 3 2

$$\gamma + X_d\tau + ², \quad \text{cov}(²) = V. \tag{1.1}$$

Here Y_d (written in column order) is the pq 1 response vector, 1_n is the n 1 column vector of ones, τ is the v 1 vector of treatment effects, X_d is a pq v plot-treatment design matrix that defines the allocation of treatments to the experimental units according to the design d, and ρ and γ are vectors of parameters for fixed row and fixed column effects, respectively. The matrices $Z_1 = 1_q \otimes I_p$ and $Z_2 = I_q \otimes 1_p$ are called the plot-row and plot-column incidence matrices, respectively. The error co-variance matrix is assumed here to be a special case ($\alpha = \beta$) of the following doubly geometric process :

$$V = \frac{\sigma^2(1-\alpha^2)^{\ 1}}{(1-\beta^2)} \begin{pmatrix} 1 & \beta & \beta^2 & \dots & \beta^{q\ 1} \\ \beta & 1 & \beta & \dots & \beta^{q\ 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta^{q\ 1} & \beta^{q\ 2} & \beta^{q\ 3} & \dots & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{p\ 1} \\ \alpha & 1 & \alpha & \dots & \alpha^{p\ 2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{p\ 1} & \alpha^{p\ 2} & \alpha^{p\ 3} & \dots & 1 \end{pmatrix}.$$

With $Z = (1_{pq} \; Z_2 \; Z_1)$, the generalized least squares information matrix C_d for estimation of treatment contrasts under (1.1) can be written as

$$C_d = X$$

is universally optimal (in the sense of Kiefer, 1975) over $D(v = 2, p, q)$ for all $\alpha \geq 0$, $\beta \geq 0$ and for all even p and q. The author is not aware of any other paper that gives optimal designs in the present set-up. It appears that even for the special case of v = 2, the optimality problem is only partially solved under the model (1.1). For example, optimal p q designs are not known for all $\alpha, \beta \in (-1, 1)$ when at least one of $p \geq 3$ and $q \geq 3$ is odd. At this time, we do not know if optimal p q designs for two treatments can be determined for all p, q, α and β mentioned above ; see the information matrix C_d in Uddin (1997) and the algebraic complexity involved in the determination of such optimal designs. However, for small p and q, the problem can be solved by enumerating all possible designs for a given p and q. In this paper, we have determined optimal 3 3, 3 4, and 3 5 designs for two treatments under (1.1) with $\alpha = \beta$. We have enumerated all possible designs in each case and determined optimal designs by comparing c_{d11} of all designs for a given p and q. Our results are presented in the following section.

2. Optimal designs for v = 2.

We have utilized MAPLE software to simplify the information matrix C_d and obtained c_{d11} for all possible $d \in D(v = 2, p, q)$ for each combination of p and q mentioned above. Note that two treatments can be assigned to pq experimental units in 2^{pq} ways, each of these arrangement is a p q design. However, not all of these designs are connected since C_d is a zero matrix for some d. In our search of optimal designs, we have calculated c_{d11} element of C_d for each connected design d. The optimal design is one that maximizes c_{d11} over $D(v = 2, p, q)$ for $\alpha \in (-1, 1)$. However, no single design is found that maximizes c_{d11} over $D(2, p, q)$ for all $\alpha \in (-1, 1)$. The optimal design depends on the magnitude of p , q and α.

In the following subsections, we use the convention that two designs d_1 and d_2 are distinct if d_1 can not be obtained from d_2 by interchanging the two symbols 1 and 2 in d_2, or d_1 can not be obtained by rotating the design d_2.

2.1 Optimal 3 3 designs for v = 2.

In this case, c_{d11} of all connected 3 3 designs are obtained using MAPLE software. We have found four distinct designs d_1, d_1', d_2, and d_3 such that the max(c_{d11}, c_{d12}, c_{d22}) is minimum.

The first four designs in Table 3 above have the same c_{d11} and hence are equally good. The values of α in column one are determined by comparing the c_{d11} values reported in third column under c_{d11}.

Note that the optimal p q design (with $p \leq q$) for two treatments, when α = 0 (errors are uncorrelated) and both p and q are odd, uses treatment one $p(q-1)/2$ times and treatment two $p(q + 1)/2$ times, see Morgan and Uddin (1993). Thus the optimal designs with uncorrelated errors require that the two treatment replications differ by p. However, this is not the case for our optimal designs with large α, see the designs in Tables 1 and 3 for large α. Here the difference between the replications of two treatments is one, a criterion often preferred by practicing statisticians.

We have determined only 3 3, 3 4 and 3 5 optimal row-column designs for two treatments. It would be unwise to make any recommendation for all p q designs based on these three designs. However, we suspect that the treatment allocation patterns found here, if extended to p q designs, will give optimal p q designs especially for large α. For example, a design in which no treatment is neighbored by itself in rows and in columns is expected to be optimal for large positive α.

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