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Abstract

It is a well known fact from the group theory that irreducible tensor representations of classical groups are suitably characterized by irreducible representations of the symmetric groups. However, due to their di!erent nature, vector and spinor representations are only connected and not united in such description.

Cli!ord algebras are an ideal tool with which to describe symmetries of multiparticle systems since they contain spinor and vector representations within the same formalism, and, moreover, allow for a complete study of all classical Lie groups. In this work, together with an accompanying work also presented at this conference, an analysis of q-symmetry – for generic q's – based on the ordinary symmetric groups is given for the Þrst time. We construct q-Young operators as Cli!ord idempotents and the Hecke algebra representations in ideals generated by these operators. Various relations as orthogonality of representations and completeness are given explicitly, and the symmetry types of representations is discussed. Appropriate q-Young diagrams and tableaux are given. The ordinary case of the symmetric group is obtained in the limit q → ". All in all, a toolkit for Cli!ord algebraic treatment of multi-particle systems is provided. The distinguishing feature of this paper is that the Young operators of conjugated Young diagrams are related by Cli!ord reversion, connecting Cli!ord algebra and Hecke algebra features. This contrasts the purely Hecke algebraic approach of King and Wybourne, who do not embed Hecke algebras into Cli!ord algebras.

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natural idea to bring the symmetric group and its deformation, the Hecke algebra, into the Cli!ord formalism. Furthermore, the symmetric group is the Coxeter group of the Dynkin diagram of the A_n

spaces which appear as a natural outcome of the embedding of the symmetric group and its representations can then be looked at as multiparticle spinor states. However, these might not be spinors of the full Cli!ord algebra.

In order to be as general as possible, we give not only the representations of the symmetric group but also of the Hecke algebra $H_F(n, q)$. The Hecke algebra is the generalization of the group algebra of the symmetric group by adding the requirement that transpositions t_i of elements $i, i+1$ are no longer involutions s_i. We set $t_i^2 = (1-q)t_i + q$ which reduces to $s_i^2 = 1$ in the limit $q \to 1$.

Hecke algebras are 'truncated' braids, since a further relation (see (3) below) is added to the braid group relations as in [4]. A detailed treatment of this topic with important links to physics may be found, for example, in [20, 34] and in the references of [11].

The deÞning relations of the Hecke algebra will be given according to Bourbaki [8]. Let $< 1, t_1, \dots, t_n >$ be a set of generators which fulÞll these relations:

$$t_i^2 = (1-q)t_i + q, \tag{3}$$

$$t_i t_j = t_j t_i, \quad i - j \geq 2, \tag{4}$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \tag{5}$$

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The bilinear form B in (8) is our particular choice that guarantees that the following equations hold:

$$\$(t_i) = b_i := e_i \wedge e_{i+n}, \tag{9}$$

$$b_i b_j = b_j b_i, \quad \text{whenever} \quad i - j \geq 2, \tag{10}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}. \tag{11}$$

This shows $\$$ to be a homomorphism of algebras implementing the Hecke algebra structure in the Cli!ord algebra $C!(B, V)$.

Notice, that the q-antisymmetrizer is related to the q-symmetrizer by the operation of reversion denoted by tilde in the Cli!ord algebra $C!_{1,1}$, that is, $C(12)^{\tilde{}} = R(12)$ and $R(12)^{\tilde{}} = C(12)$. How do we know that $q + b_1$ gives the symmetrizer $R(12)$? Notice Þrst that $R(12)$ is almost an idempotent since

$$R(12)R(12) = (1 + q)e_1 \wedge e_5 + q(1 + q) = (1 + q)R(12). \tag{14}$$

Thus, when we normalize $R(12)$ by dividing it by $1 + q$, the new element denoted as $R(12)_q$ will be an idempotent.

$$R(12)_q R(12)_q = \frac{(1 + q)e_1 \wedge e_5 + q(1 + q)}{(1 + q)^2} = \frac{e_1 \wedge e_5 + q}{1 + q} = \frac{b_1 + q}{1 + q} = R(12)_q.$$

If we now take the limit of $R(12)_q$ as $q \to 1$, we obtain

$$\lim_{q \to 1} R(12)_q = \frac{1 + s_1}{2} \tag{15}$$

with $s_1 = e_1 \wedge e_5$ squaring to 1 (in the limit $q \to 1$) in agreement with (i) above. Then the expression $\frac{1}{2}(1 + s_1)$ acts as a symmetrizer on, for example, functions of two variables. Likewise, the normalized q-antisymmrali(or)-1l qe.R

non-primitive idempotents. Each of these idempotents generates a three-dimensional decomposable ideal. To achieve this goal, we can use any two of the following three equations since any two of them imply the third:

$$
\begin{aligned}
X + \tilde{X} &= 1, && (24)\\
X^2 &= X, && (25)\\
X\tilde{X} &= 0. && (26)
\end{aligned}
$$

By doing so, our goal is to Þnd four Young operators known to exist from the general theory of the Hecke algebras for $n = 3$ [20, 34]. The four q-Young operators will still have only one parameter and they will generalize four Young operators of S_3 described in Hamermesh [21] on p. 245. One of them will be a full symmetrizer, another one will be a full antisymmetrizer, and the other two will be of mixed symmetry.

In the Þrst step, we will Þnd the most general element

$$X = K_1 1 + K_2 b_1 + K_3 b_2 + K_4 b_{12} + K_5 b_{21} + K_6 b_{121} \tag{27}$$

in the Hecke algebra $H_F(3,q)$ that satisÞes (24). Upon substituting X into (24) we have found that X must have the following form:

$$
\begin{aligned}
X \;=\; & (\tfrac{1}{2}qK_2 - \tfrac{1}{2}qK_6 + \tfrac{1}{2} + \tfrac{1}{2}qK_3 - \tfrac{1}{2}K_2 + \tfrac{1}{2}q^2K_6 - \tfrac{1}{2}K_3)1 \\
& + K_2 b_1 + K_3 b_2 + K_4 b_{12} + (-K_4 + qK_6 - K_6) b_{21} + K_6 b_{121}.
\end{aligned} \tag{28}
$$

The element X in (28) belongs to a family parameterized by four real or complex parameters K_2, K_3, K_4, K_6. Next we demand that X also satisÞes (25).

After substituting X displayed in (28) into equation (25), we have found six sets of solutions. The solutions are parameterized by complex numbers satisfying two similar but di!erent quadratic equations:

$$
\begin{aligned}
(1+q)z^2 + (-q^2K_4 + K_4 + qK_2 - 1 + K_2)z + K_4K_2 + K_2^2 - K_4 - K_2 \\
+qK_4 - q^2K_4^2 - qK_4^2 - q^2K_2K_4 + qK_2^2 = 0,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
(1+q)z^2 + (-q^2K_4 + K_4 + qK_2 + 1 + K_2)z + K_4K_2 + K_2^2 + K_4 + K_2 \\
-qK_4 - q^2K_4^2 - qK_4^2 - q^2K_2K_4 + qK_2^2 = 0.
\end{aligned} \tag{30}
$$

$$r_3 = \frac{1}{1+q} + qK_4b_1 - K_4b_2 + K_4b_{12} - \frac{(q^3K_4 + q + K_4 - 1)b_{21}}{q(1+q)} - \frac{(-K_4 + q^2K_4 + 1)b_{121}}{q(1+q)},$$

$$r_4 = \frac{q1}{1+q} + qK_4b_1 - K_4b_2 + K_4b_{12} - \frac{(q^3K_4 - q + K_4 + 1)b_{21}}{q(1+q)} - \frac{(-K_4 + q^2K_4 - 1)b_{121}}{q(1+q)},$$

$$\begin{aligned} r_5 = {} & \frac{1}{1+q} + K_2b_1 + K_4b_{12} \\ & - \frac{(' + K_2 - qK_4 + K_4 - q^2K_4^2 + qK_2^2 - q^2K_2K_4 - qK_4^2 + K_2^2 + K_4K_2)b_2}{(K_2 + K_4 - qK_4 + ')(1+q)} \\ & - \frac{(q'K_4 + qK_2K_4 + q'K_2 - 'K_2)b_{21}}{q(K_2 + K_4 - qK_4 + ')} + \frac{('K_2 + qK_4^2)b_{121}}{q(-K_2 - K_4 + qK_4 - ')}, \end{aligned}$$

$$\begin{aligned} r_6 = {} & \frac{q1}{1+q} + K_2b_1 + K_4b_{12} \\ & - \frac{(q^2K_4^2 - qK_2^2 + q^2K_2K_4 + qK_4^2 + \& - qK_4 + K_2 + K_4 - K_2^2 - K_4K_2)b_2}{(-K_2 - K_4 + qK_4 - \&)(1+q)} \\ & + \frac{(qK_2K_4 + q\&K_4 + q\&K_2 - \&K_2)b_{21}}{(-K_2 - K_4 + qK_4 - \&)q} + \frac{(\&K_2 + qK_4^2)b_{121}}{q(-K_2 - K_4 + qK_4 - \&)}. \end{aligned}$$

It can be checked with CLIFFORD that the rank of the set r_i , $i = 1, \ldots, 6$, is four. For our purpose we must select any four linearly independent elements, for example r_1, r_2, r_3, r_5 , which we rename f_1, f_2, f_3, f_4 . It can also be veriÞed with CLIFFORD that the elements f_1, f_2, f_3, f_4 satisfy the required relations (24), (25), and (26).

We look for the Young operators obtained by one of the $f_i, i = 1, \ldots, 4$. Due to the fact that the representation spaces which correspond to the symmetric $Y^{(3)}_{1,2,3}$ and the antisymmetric $Y^{(111)}_{1,2,3}$ Young operators respectively are one-dimensional, they cannot have any free parameters besides q. The full symmetrizer can be given according to KW as the q-weighted sum of all six Hecke basis elements. However, in our construction the full antisymmetrizer is deÞned as the reversion of the full symmetrizer, that is, $Y^{(111)}_{1,2,3} := Y^{(3)}_{1,2,3}\tilde{}$, as it was done in dimension two. Then we have:

$$Y^{(3)}_{1,2,3}$$

of the f_i elements has to be a sum of a full (anti)symmetrizer and a Young operator of the mixed type. If we pick f_1 we notice that it must contain the full antisymmetrizer, because when the parameter K_4 is replaced with $1/(q+1)$ then the re-deÞned f_1 (or the r_1 deÞned above) reduces to an expression with alternating signs in the Hecke basis:

$$\frac{1}{1+q} - \frac{b_1}{1+q} + \frac{qb_2}{1+q} + \frac{b_{12}}{1+q} - \frac{qb_{21}}{1+q} - \frac{b_{121}}{1+q}.$$

Therefore, by subtracting the full antisymmetrizer $Y^{(111)}_{1,2,3}$ from f_1 we Þnd our Þrst Young operator $Y^{(21)}_{1,3,2}$ of the mixed type:

$$\begin{aligned} Y^{(21)}_{1,3,2} &= f_1 - Y^{(111)}_{1,2,3} \\ &= \frac{q1}{q+1+q^2} - \frac{(q^3K_4 + 2q^2K_4 + 2qK_4 - 1 + K_4)b1}{q^3+2q^2+2q+1} \\ &\quad + \frac{(K_4q^4 + 2q^3K_4 + 2q^2K_4 + qK_4 + 1)b_2}{(q^3+2q^2+2q+1)} \\ &\quad + \frac{(q^3K_4 + 2q^2K_4 + 2qK_4 - 1 + K_4)b_{12}}{(q^3+2q^2+2q+1)} \\ &\quad - \frac{(K_4q^5 + K_4q^4 + q^3K_4 + q^3 + q^2K_4 + qK_4 + q + K_4 - 1)b_{21}}{(q^3+2q^2+2q+1)q} \\ &\quad - \frac{(K_4q^4 + q^3K_4 + q^2 - qK_4 + 1 - K_4)b_{121}}{(q+1+q^2)q(1+q)}. \end{aligned} \tag{33}$$

We deÞ

which decompose the unity in the Hecke algebra since $f_1 + \tilde{f}_1 = 1$. It can be easily veriÞed that the Young operators of mixed type decompose into the row-symmetrizer and the colum-antisymmetrizer in accordance to Hamermesh [21] p. 245. Our expressions however are di!erent from those in KW.

In order to represent our Young operator $Y^{(21)}_{1,3,2}$ as a product of the row symmetrizer $R(13)$ and the column antisymmetrizer $C(12)$, we use previously deÞned $f_1 = r_1$ to deÞne $C(12) := f_1$ and compute $R(13)$ from the equation

$$Y^{(21)}_{1,3,2} = R(13) f_1. \tag{36}$$

Notice that our $f_1 = r_1$ is a generalization to S_3 of $C(12)$ from S_2 displayed in (13). In an e!ort to be consistent with our previous discussion of $C(12)$ and $R(12)$, which were related by the reversion, we will later deÞne $C(13) := R(13)\tilde{}$ and require that $R(13) + C(13) = R(13) + R(13)\tilde{} = 1$. Thus, when we solve (36) for $R(13)$, we get the following solution:

$$R(13) \quad = \quad \frac{q1}{1+q} - \frac{(-q^2 + q^2 P_3 + P_3 q - 1 + P_3)b_1}{(}$$

parameterized only by K_4 as follows:

$$T = \frac{(1 - K_4 - q^3 - K_4 q + K_4 q^3 + q^4 K_4)1}{}$$

With CLIFFORD it has been veriÞed that the equation (43) is satisÞed automatically by each $X X_i$, $i = 1$

where t_6, t_7, t_8 are polynomials.[5] With CLIFFORD we have veriÞed that the six elements in the list S below are linearly independent. As such, they provide a basis for the left regular representation of the Hecke algebra:

$$S = [Y^{(3)}_{1,2,3}, Y^{(21)}_{1,}$$

$$M_{b_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{qp_5}{1+q} & \frac{p_{16}}{1+q} & 0 & 0 & 0 \\ 0 & -\frac{qp_6}{p_{17}} & \frac{p_{21}}{1+q} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{qp_{22}}{p_4} & \frac{-q^3}{p_3} & 0 \\ 0 & 0 & 0 & -\frac{p_{23}}{qp_4} & -\frac{p_7}{p_4} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q \end{pmatrix} \tag{55}$$

Matrices M_{b_1} and M_{b_2} satisfy, of course, the same quadratic Hecke relation (3) as do b_1 and b_2, and which happens to give the minimum polynomial $p(x) = x^2 - (1-q)x$

constructed in the paper corresponding to the Young tableaux conjugate to each other in the sense of Macdonald and which generate representation spaces of dimension greater than one, have been related through the reversion in the Cli!ord algebra $C\ell_{4,4}$. This feature is not present in King and Wybourne.

In $H_F(2,q)$ we found that the symmetrizer $R(12)$ and its reverse, the antisymmetrizer $C(12)$ were primitive idempotents in the Cli!ord algebra. We found no nontrivial intertwiners linking these two idempotents.

In $H_F(3,q)$ we Þrst found four mutually annihilating idempotents splitting the unity in the algebra: two without parameters and two parameterized ones. The Þrst two were the Young symmetrizer $Y^{(3)}_{1,2,3}$, deÞned as in King and Wybourne, and the Young antisymmetrizer $Y^{(111)}_{1,3,2}$, deÞned in this paper as the reverse of $Y^{(3)}_{1,2,3}$. The two parameterized idempotents were the Young operators of mixed symmetry $Y^{(21)}_{1,2,3}$ and $Y^{(21)}_{1,3,2}$, and they were also related by the reversion. We were able to factor $Y^{(21)}_{1,3,2}$ into the row symmetrizer $R(13)$ and the column antisymmetrizer $C(12)$, that is, we were able to Þnd $R(13)$ as an idempotent element in the Cli!ord algebra, a feature not found in King and Wybourne. Furthermore, we related the mixed-type Young operators through a Þve-parameter family of intertwiners.

We have found a Garnir element $G^{(21)}$

Appendix

Polynomials below have been introduced as abbreviation to improve readability of the formulas displayed in the main text:

$$
\begin{aligned}
p_1 &= q^4K_4^2 + 3q^3K_4^2 - K_4q^3 + 4q^2K_4^2 - K_4q^2 + 3qK_4^2 \\
&\quad -K_4q - q + K_4^2 - K_4, \\
p_2 &= K_4q^4K_6 + q^4K_4^2 + q^3K_4K_6 - q^3K_4K_2 + K_4q^2 \\
&\quad -q^2K_2K_4 + q^2K_4K_6 + K_6q^2 + q^2K_4K_5 - q^2K_4^2 \\
&\quad -K_2q - K_6q + qK_4K_5 - qK_4^2 + K_4qK_6 - K_5q \\
&\quad -qK_2K_4 - K_4q - K_4^2 - K_4K_2 + K_4 + K_2, \\
p_3 &= -K_6q^3 - K_4q^3 + K_6q^2 + K_5q^2 + K_6q + K_5q - K_6 - K_4, \\
p_4 &= -K_6q^2 - K_4q^2 + 2K_6q + K_5q + K_4q - K_6 - K_4, \\
p_5 &= K_4q^3 + 2K_4q^2 + q + 2K_4q + K_4, \\
p_6 &= q^5K_4^2 + 3q^4K_4^2 + 4q^3K_4^2 + K_4q^3 + 3q^2K_4^2 \\
&\quad +K_4q^2 + qK_4^2 + K_4q + K_4 - 1, \\
p_7 &= -K_4q^3 - q^3K_2 + 2K_6q^2 + K_5q^2 + K_4q^2 - 2K_6q - K_5q \\
&\quad -K_4q + K_6 + K_4, \\
p_8 &= q^5K_4 + q^4K_4 + q^3 + K_4q^3 - q^2 + K_4q^2 + K_4q + K_4 - 1, \\
p_9 &= q^6K_4^2 + 2q^5K_4^2 + 2q^4K_4^2 + 2q^4K_4 + 2q^3K_4^2 \\
&\quad +K_4q^3 + 2q^2K_4^2 + q^2 + 2qK_4^2 - K_4q - q + K_4^2 - 2K_4 + 1, \\
p_{10} &= q^3K_2 - K_6q^3 + K_6q^2 - q^2K_2 + K_5q + K_2q - K_5 - K_6, \\
p_{11} &= q^4K_4^2 + 3q^3K_4^2 + 4q^2K_4^2 + K_4q^2 + 3qK_4^2 \\
&\quad +K_4^2 - K_4 + 1, \\
p_{12} &= q^5K_4^2 + 2q^4K_4^2 + q^3K_4^2 + 2K_4q^3 - q^2K_4^2 \\
&\quad +2K_4q^2 - 2qK_4^2 + 2K_4q + q - K_4^2 + 2K_4 - 1, \\
p_{13} &= q^4K_4 + K_4q^3 + q^2 - K_4q - K_4 + 1, \\
p_{14} &= q^2K_2 - K_6q^2 + K_6q - K_2q - K_4 + K_5, \\
p_{15} &= K_4q^3 + 2K_4q^2 - q^2 + 2K_4q + K_4, \\
p_{16} &= 3K_4q^4K_6 + 3q^3K_4K_6 + K_4qK_{6=}
\end{aligned}
$$

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