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TERTIARY MATHEMATICS EDUCATION RESEARCH AND ITS FUTURE

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Tertiary Mathematics Education Research and Its Future

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1. What is Mathematics Education Research at the University Level?

Tertiary mathematics education research is disciplined inquiry into the learning and teaching of mathematics at the university level. It can be conducted from an individual cognitive perspective or from a social perspective of the classroom or broader community.[1] It can also coordinate the two, providing insight into how the psychological and social perspectives relate to and impact one another.

In the case of individual cognition, one wants to know how students come to understand aspects of mathematics, limit or linear dependence, or how they develop effective mathematical practices, good problem-solving skills, the ability to generate reasonable conjectures and to produce proofs. What goes on in students' minds as they grapple with specific, i.e., often concerns questions directly involving the understanding of

beyond and leads to emphasizing an individual cognitive perspective. In contrast, those taking a sociocultural view, emphasize the idea that culture mediates individual knowledge through tools and language, an idea that has roots in Vygotsky's work and leads to taking a social perspective.[4] However, such philosophical views apply to all knowledge, not just to mathematics and are often not particularly conspicuous in the research findings themselves.

Mathematics education research, being domain specific, has developed its own ways of conceptualizing mathematics learning. Indeed, Robert B. Davis, who was a long-time editor of the *Journal of Mathematical Behavior,* pointed out that one of the major contributions of mathematics education research has been to provide new conceptualizations and new metaphors for thinking about and observing mathematical behavior (Davis, 1990). It is very difficult to notice patterns of behavior or thought without having names (and the corresponding concepts) for them. As is often said of other empirical disciplines, one needs a lens (theoretical framework) with which to focus on (frame) what one is seeing.

2.1.1 Concept Definition versus Concept Image

The mismatch between concepts as stated in definitions and as interpreted by students is well-known to those who teach university level mathematics. The terms *concept definition* and *concept image* were introduced into the mathematics education literature to distinguish between a formal mathematical definition and a person's ideas about a particular mathematical concept, such as function. An individual's concept image is a mental structure consisting of all of the examples, nonexamples, facts, and relationships, etc., that he or she associates with a concept. It need not, but might, include the formal mathematical definition and appears to play a major role in cognition. [Cf. Tall & Vinner, 1981.] Furthermore, while contemplating a particular mathematical problem, it might be that only a portion of one's concept image, called the *evoked concept image*, is activated. These ideas make it easier to understand and notice various aspects of a student's thinking, for example, a student who conceives of functions mainly graphically or mainly algebraically without much recourse to the formal definition. A teacher or a researcher can investigate what sorts of activities might encourage students to employ the definition when that is the appropriate response, as in making a formal proof. It might also be helpful to investigate how students develop their concept images or how such images affect problem-solving performance.

2.1.2 Obstacles to Learning

One set of related ideas that has proved powerful is that of *epistemological, cognitive, and didactic obstacles*.[5] When applied to the learning of mathematics, these refer, respectively, to obstacles that arise from the nature of particular aspects of mathematical knowledge, from an individual's cognition about particular mathematical topics, or from particular features of the mathematics teaching. An obstacle is a piece of, not a lack of, knowledge, which produces appropriate responses within a frequently experienced, but limited context, and is not generalizable beyond it (Brousseau, 1997).

A further idea that appears to be very useful, but which may not yet be widely found in the literature, is the distinction between *synthetic* and *analytic definitions*.[7] Synthetic definitions are the everyday definitions that are commonly found in dictionaries -- they are descriptions of something that already exists. They are often incomplete, yet redundant. For example, on the crudest level, one can define a democracy as a form of government in which the people vote. However, additional properties might better characterize governments normally regarded as democracies. It is often unclear when such everyday definitions are "complete" or whether attention to all aspects of them is essential for their proper use. Analytic definitions, by contrast, bring concepts into existence -- the concept is whatever the definition says it is, nothing more and nothing less. Thus, for example in a graduate course, one usually defines a semigroup as a set together with a binary associative operation on it and immediately begins to deduce properties about semigroups. One cannot safely ignore any aspects of such definitions. Many of the difficulties that university students have with formal mathematics might well be viewed as stemming partly from an unawareness that mathematical definitions tend to be analytic, rather than synthetic, or from an inability to handle formal mathematics even when a difference in the two kinds of definitions is perceived.

2.1.5 How Might These Concepts Might be Used by University Teachers of Mathematics?

Such ideas (e.g., concept image, epistemological obstacle, action-process-object-schema, synthetic vs. analytic definitions) help frame, not only research, but also discussions of teaching and learning toward more insightful, and ultimately, more productive ends. They help one view students' attempts at mathematical sense-making and understanding as somehow hindered by their current, somewhat limited, conceptualizations -- instead of merely emphasizing that university students don't do their homework, aren't motivated, or are just plain lazy (some of which may also be true).

While it is perhaps too soon to expect such ideas to have moved far beyond the mathematics education research community and it is hard to gauge the practical effects of anyone's use of new concepts, there are a few hints that some teachers and authors are finding them useful. For example, not long ago an author of undergraduate mathematics textbooks indicated that he finds the idea of concept image useful in his teaching and writing.

In general, the pedagogical challenge is to figure out how to help students come to genuine mathematical understanding. Which instructional efforts might be more productive of genuine mathematical understanding? Here various techniques have been tried -- computer activities that provoke students to reflect on mathematical situations (and to explicitly construct actions, processes, objects, and schemas), group projects that require them to grapple with mathematical ideas, and process writing to help students clarify their mathematical thoughts (by explicitly describing the evolution of their thinking whilst wrestling with problem situations). For example, when there was concern about the pass rate of U.S. university calculus students, the National Academy of Science convened a National Symposium in 1987 which resulted in a concerted effort on the part of the National Science Foundation to promote calculus reform.[8] Some combination of the above pedagogical strategies was included in many of the resulting calculus reform projects (Tucker, 1990), but very few of these were based on research ideas such as those mentioned above.

2.2 Some Theories of Instructional Design

While the above ideas can prove helpful to university teachers of mathematics desiring to understand "where their students are coming from," it would also be useful for mathematics education research to inform the development of curricula. Although there are relatively few mathematics education researchers working at the university level worldwide and information is really just beginning to accumulate, there have been some efforts at curriculum design using the results obtained so far. Here are four examples, the last of which did not arise from the research literature, but is in considerable agreement with it. All four teach through student-solved problems and avoid providing worked, template examples.

2.2.1 APOS Theory and the ACE Teaching Cycle

The learning of many university level mathematical topics has been investigated using the APOS (Action-Process-Object-Schema) theory and instructional sequences have been designed reflecting it. Envisioned as an iterative process, this instructional design process begins with a theoretical analysis, called a *genetic decomposition*, of what it means to understand a concept and how that understanding might be constructed or arrived at by the learner. This initial analysis is based on the researchers' understanding of the concept and on their experiences as learners and teachers of mathematics. This leads to the design of instruction, which is subsequently implemented and observed. Data is gathered and analyzed, and this analysis leads to revisions of both the theoretical analysis and the instructional design.

Since this approach views the growth of mathematical understanding as highly non-linear -- with students developing partial understandings, often repeatedly returning to the same concept -- the instructional approach consists of "an *holistic spray*, a variation of the standard spiral method." Students are intentionally put into disequilibrating situations (in which they see their lack of understanding) and, individually or in cooperative groups, they try to make sense of these situations, e.g., by solving problems, answering questions, or understanding ideas. A particular strategy used is the *ACE Teaching Cycle*, consisting of three components: Activities, Class discussion, and Exercises. The activities often involve extensive teamwork on ISETL[9] computer programming tasks, whose design is based on the proposed genetic decomposition of a particular concept. The intent is to provide students with experiences that promote the development of that concept and upon which they can build in the forthcoming discussions. In the instructor-led class discussions which usually take place on a subsequent day, the students again work in teams, but this time on paper-and-pencil tasks based on the computer activities. This is followed by relatively traditional out-of-class exercises to be worked individually or in teams; their purpose is to reinforce the mathematics learned. Because this pedagogical strategy is somewhat unconventional, its designers have found it necessary to create textbooks to support it; this has been done for discrete mathematics, precalculus, calculus, and abstract algebra. These textbooks do not contain template problems and no level, is that of *realistic mathematics education*. From this perspective, students learn mathematics by mathematizing the subject matter through examining "realistic" situations, i.e., experientially real contexts for students that draw on their current mathematical understandings. In this approach, the problems precede the abstract mathematics, which emerges from the students' collaborative work towards solutions. This approach goes back to Freudenthal and is favored by the Dutch school of mathematics education researchers. Curricula, as well as the instructional theory and its justification, are mutually developed and refined in a gradual, iterative process.

In this approach, curriculum design tends to be integrated with research, perhaps because it is difficult to predict how students will tackle problems for which they have no model solutions. Beginning with realistic mathematics education as the global perspective, the aim is to develop *local instructional theories*, whether these be for the teaching and learning of fractions or differential equations. In a manner somewhat analogous to the cycle of development mentioned in 2.2.1, this developmental process begins by positing hypothetical learning trajectories, along with a set of instructional activities. After an instructional sequence has been implemented and observed, researchers engage in retrospective analysis that leads to refinement and revision of the conjectured learning trajectory. Three heuristics are used in designing curricula: (1) the *reinvention principle*, whereby students are guided to construct at least some of the mathematics for themselves, (2) *didactic phenomenology*, whereby researchers analyze practical problems as possible starting points for the reinvention process, and (3) the construction of *mediating, or emergent models* of students' informal knowledge and strategies in order to assist students in generalizing and formalizing their informal mathematics. [Cf. Gravemeijer, 1998; **Note to the editors: Here there could also be a cross-reference to the conference paper of Chris Rasmussen and Karen King.]**

2.2.4 The Moore Method of Teaching

This distinctive method of teaching has developed into an informal method of curriculum design and has evolved naturally without calling on research or theory in mathematics education. However, although it arose prior to, and independently of, didactical engineering and the work of Brousseau, some of its aspects are derivable from that work. There is a renegotiation of the didactic contract, a teaching through carefully selected problems (usually requiring the construction of proofs), "devolution" of the problems to the students (i.e., transferring to them an interest in, and an obligation for, the production of proofs), and "adidactic situations" requiring the students to solve the problems on their own. Students are presented many opportunities (situations) for facilitating personal knowledge construction, but also construct much of the actual mathematics, usually in the form of proofs, themselves.

The method developed out of the teaching experiences of a single accomplished U.S. mathematician, R. L. Moore, and has been continued by his students (several of whom went on to become presidents of the American Mathematical Society or the Mathematical Association of America) and their mathematical descendents. It has been remarkably successful in producing research mathematicians, but has also been used in undergraduate university classes. In many versions, students are given definitions and statements of theorems or conjectures and asked to prove them or provide counterexamples. The teacher provides the structuring of the material and critiques the students' efforts. Since classroom in order that students will come to see revised classroom practices, such as the necessity to give reasons, as "normal" and also build more interest in mathematics itself.

(3). Most experienced undergraduate mathematics teachers can easily identify a number of topics with which many students will have difficulties. These include the concept of variable, working with "split domain" functions, limits, the Fundamental Theorem of Calculus, sequences and series, the ideas of proof and vector space, and the open cover definition of compactness. Methods arising from APOS theory are particularly concerned with helping students construct mathematical concepts and are likely to be helpful in teaching such topics.

(4). Finally, it is probably an annoyance to many teachers of tertiary mathematics that their students, and indeed most of the general public, have very little idea what mathematics is about. This is not just a matter of inco4[ctions, ess. 4 with

review (referee) several manuscripts, thereby introducing them to the criteria for acceptance. Normally, editors try not to overburden individuals, but reviewing papers can be very educational, especially where reviewers are ultimately provided with the editor's and other reviewerseaactice for the *Journal for Research in Mathematics Education*.

Also, more mentoring programs, like those of the RUMEC group and ARUME might be beneficial. One or two-daysh19 courses, such as those g4pheriven at meeting4phers of the American Mathematical Society (MS) and the Mathematical Association of America (MAA), mig4pherht be productive. In order to join a research team, as in the French model, individuals usually need financial supn19. Often, one's home university, will not finance extended full-time leaves. One promising prog41.1(r)4.1(a)4.9(m in the)4.9(U.S. is the)4.9(Na)4.9(tiona)4. supervised by an appropriate mentor at a major university.

3.3. The Placement of Tertiary Mathematics Education Researchers

Where will tertiary mathematics education researchers find their academic “home”? Althouge are a number of Ph.D.-grantingcing researchers in tertiary mathematics education, unlesD-0.1fny mathechnologyg

and promotion. It might be possible to alleviate the effect of this dearth of high quality exposition by establishing more links between existing websites.

Other avenues for dissemination include the annual meetings of ARUME with MAA, at which mathematics education research papers will be featured, along with an expository talk. However, at best this effort can only reach the few thousand mathematicians that attend such meetings. A similar role might be played in the U.K., by the Advanced Mathematical Thinking Working Group. Another modest start towards dissemination and recognition of the field, is the fact that *Zentralblatt für Didaktik der Mathematik (ZDM)* and *Mathematical Reviews (MR)* now both include coordinated categories for abstracting research articles in undergraduate mathematics education.

It would be especially beneficial to find ways to bring tertiary mathematics education research results to the attention of graduate students in mathematics, many of whom will take up teaching posts in universities. In the U.S., the Exxon-funded Project NEXT (New Experiences in Teaching) has given several hundred new mathematics faculty members the opportunity to meet and network at annual meetings of MAA, while also attending special workshops on technology and teaching that include some mention of tertiary mathematics education research results.

4.2 Integrating Research Results into Teaching Practice

The most effective way of bringing tertiary mathematics education research into teaching practice, seems to be via new research-based curricula. In Section 2.2 there are three examples of ways that research has been systematically integrated with curriculum design. In addition, the results of research can also be used in less systematic ways to inform teaching and curriculum development. For example, mathematics education researchers at San Diego State University in the U.S. have used research results in preparing a CD-ROM to assist university instructors of mathematics courses for pre-service elementary and middle school mathematics teachers. Several other such video/CD-ROM projects exist, but it is not clear whether these are targeted at those teaching in mathematics departments or in education departments. In general, however, the mathematics community has yet to make much use of research results in either teaching or curriculum design. Thus the need for enhanced dissemination in order to bring about long-range benefits.

4.3 Some Suggestions for Reaching University Mathematics Teachers

Perhaps team teaching, departmental seminars on teaching, or other local efforts could facilitate the incorporation of research results and generally improve pedagogy. One http://www2.admin.ias.edu/ma/park.htm.] Such workshops and institutes tend to be expensive; the two mentioned here were funded by the U.S. National Science Foundation (NSF).

Currently, there are "research into practice" sessions (Wilson, 1993) at NCTM meetings; perhaps similar sessions at meetings of university mathematics teachers, such as those of the AMS/MAA, would be beneficial. These might include video clips from research, showing students engaged in mathematical problem solving, or teaching episodes; such clips often provide convincing visual evidence and "talking points" for interested, even skeptical, teachers.

It is important to have mathematics Ph.D. students, who will become tomorrow's university mathematics teachers, take research in tertiary mathematics education seriously. Perhaps, in those Ph.D. programs where coursework is required, one could insist that students take a course on tertiary mathematics education research, or even conduct a mini-research project on some aspect of students' mathematical thinking. One practical problem is: Who would teach such a course? Also, perhaps it would be helpful to develop a list of expository and other readings in tertiary mathematics education research, post it on the Web, and update it regularly.[17] If so, whose responsibility would that be?

5. Possible New Directions for Mathematics Education Research at the University Level

Clearly, the existing tertiary mathematics education research barely "scratches the surface." While some topics of interest in both secondary and tertiary teaching, like the function concept, have benefited from being considered by a number of researchers[18], other topics such as real analysis or the learning of post-graduate students in mathematics are just now beginning to be studied.

One fairly natural way to collect and refine research questions is to examine one's own teaching; for that to be successful, mathematics education researchers need to teach, not only preservice teachers as is often the case in the U.S., but also specialist and nonspecialist (e.g., engineering and business) students at the tertiary level.

Many areas such as students' learning, teaching and teacher change, problem solving and proofs, social structures like departments, views of mathematics, theoretical frameworks, and pedagogical content knowledge are ripe for further investigation.

5.1 Some Ideas that Might be Worth Pursuing

In his plenary address, Hyman Bass pointed to four areas of mathematics education, with some associated questions, that are critically in need of systematic research: the secondary/tertiary transition, instructional use of technology, university-level teaching, and the context of the university with respect to teaching. **[Note to editors: Here it would be good to cross-reference the plenary of Bass]**. Here is a potpourri of additional questions.

5.1.1 Beginning University Students: The Secondary/Tertiary Interface

In the U.S., many students entering junior colleges and comprehensive state universities are unprepared to take calculus, and much teaching occurs at the precalculus level. Some of these students are older, non-traditional students whose secondary mathematical preparation needs renewal. Would it be useful to catalogue the many, and possibly interacting, difficulties of these precalculus students? Lately, while teaching such courses, we noticed students who fell asleep during an examination, spent significant time murmuring "I hate math," and had difficulty reading and interpreting test questions, e.g., completely ignoring adjectives like "positive." In addition, some students are very weak in simple algebraic skills and cannot make "multi-level" substitutions. Although a idea is consistent with the success of the Moore method of teaching (described in 2.2.4), in which students invent proofs of a carefully structured sequence of statements, but are not explicitly or separately taught the prerequisite knowledge before starting to construct proofs. With the aid of hypertext, one might integrate, in a nonlinear way, learning to construct proofs with "just in time" knowledge of logic, functions, sets, etc.[19]

5.1.3 Teaching Service Courses for Non-specialists

As Artigue points out, much research at the tertiary level has, often implicitly, taken the view that universities train future mathematicians, whereas a large amount of university mathematics teaching occurs in "service courses" for "client disciplines," a trend that may well increase. **[Note to the editors: Here there might be a cross-reference to Lynn Steen's plenary.]**

There have been a few studies of how practicing professionals -- architects, biologists, bankers, nurses -- use mathematics, with the ultimate aim of improving the teaching of such courses. The classic view of mathematical modeling, which involves identifying and simplifying a problem, solving a decontextualized mathematical version, and mapping the solution back, does not agree with workplace experience. [Cf. Pozzi, Noss & Hoyles (1998); Smith, Haarer & Confrey, (1997); Smith & Douglas, (1997); Noss & Hoyles (1996)] More workplace studies of mathematics use, especially as they relate to curriculum development, would be helpful.

In addition, the teaching of mathematics to preservice elementary and secondary teachers comprises another large share of the courses taught in some mathematics departments. Because of its effect on the teaching of school level mathematics this aspect of tertiary mathematics education has been somewhat better studied and is often the subject of papers at conferences, such as those of PME and PME-NA.[20]

5.1.4 Aspects of Teaching Practice and Institutions that Affect Learning

What views of learning do tertiary mathematics teachers have and how do these affect their practice? Does a teacher's pedagogical knowledge closely resemble the kind of automated procedural knowledge that might be called upon in actual teaching practice? That is, can one predict, and ultimately change, moment-to-moment pedagogical strategies? (Schoenfeld; 1998.) In athletics, knowing how a game should be played is rather different from being able to play it.

The relationship of teaching practice and the mathematical and pedagogical beliefs of mathematics department members to the leadership and power structure of a department has not been well examined. Yet individual teacher change may depend significantly on such social structures. For example, we know a mathematician, tenured in a small department, who will try new teaching techniques on upper-division courses, but not on calculus because colleagues might disapprove.

Could some research be directed towards generating pedagogical content knowledge, e.g., how to teach the Chain Rule or an explanation of why some university students persist in adding fractions incorrectly? Such knowledge can be a major part of the preservice teacher curriculum, but there is a dearth of it at the tertiary level. Perhaps some mathematicians would be interested in discovering and analyzing pedagogical content knowledge by conducting small teaching experiments, thereby making a contribution without having to delve deeply into the theoretical aspects of mathematics education research.

5.1.5 Philosophical and Theoretical Questions

Views of mathematics arising from the current philosophical climate tend to treat mathematics as asocial or mental construct, and sometimes equate objectivity with social agreement (Ernest, 1998). This appears inconsistent with the ideas of many mathematicians who often see themselves as approaching some kind of abstract knowledge which is independent of time and place. Is there a synthesis of these two apparently contradictory positions which is compatible with both?

What are some promising directions to develop or extend theoretical frameworks? The action, process, object, schema (APOS) theory (Asiala, et al, 1996; Sfard, 1991) may not yet have reached its full potential. Concepts can be not only objects (e.g., topological spaces), but also properties (e.g., compactness) and activities (e.g., factoring). Are the last two of these learned in a way similar to the first? Or, to take another example, the classification of memory as long-term, short-term, and working seems a somewhat neglected, but promising framework. Might errors that students sometimes refer to as "dumb mistakes" be explainable in term(to)-1reovre tho(to)-1tdm is cc[(")7.7 tl.3(dagvelop ne06 ms a som

the paper of Kent & Noss in these volumes.] Research questions include: How does the

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[1] There does not seem to be any one universally accepted approach to research in mathematics education. However, since the days of Frances Bacon, science has been seen as consisting of making observations, reporting them to others, formulating theories, and finally testing them. Some mathematics education researchers emphasize the beginning of this process and see research as "disciplined noticing;" they might point out new phenomena that lead to new theoretical frameworks. Others emphasize the end of the process; using theoretical frameworks, they formulate hypotheses and test them, perhaps with teaching experiments. Furthermore, many researchers today are eclectic, adjusting their approach to the particular situation they are studying.

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