DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

# DO CALCULUS STUDENTS EVENTUALLY LEARN TO SOLVE NON-ROUTINE PROBLEMS

Annie Selden, John Selden, Shandy Hauk, Alice Mason

May 1999

No. 1999-5

TENNESSEE TECHNOLOGICAL UNIVERSITY
Cookeville, TN 38505

# Do Calculus Students Eventually Learn to Solve Non-routine Problems?

ANNIE SELDEN, JOHN SELDEN, SHANDY HAUK, ALICE MASON

ABSTRACT. In two previous studies we investigated the non-routine problem-solving abilities of students just finishing their first year of a traditionally taught calculus sequence. This paper[1] reports on a similar study, using the same non-routine first-year differential calculus problems, with students who had completed one and one-half years of traditional calculus and were in the midst of an ordinary differential equations course. More than half of these students were unable to solve even one problem and more than a third made no substantial progress toward

equipped with information and skills and is confronted with a cognitively non-trivial task; that is, the solver does not already know a method of solution. Seen from this perspective, a problem cannot be solved twice by the same person, nor is a problem independent of the solver's background. In traditional calculus courses most tasks fall more readily into the category of exercise than problem. However, experienced teachers can often predict that particular tasks will be problems for most students in a particular course, and tasks that appear to differ only slightly from traditional textbook exercises can become problems in this sense.

In this study we used the same two tests as before: a five-problem non-routine test and a ten-question routine test (see Sections 2.3.1 and 2.3.2). The second, routine, test was intended to assess basic skills sufficient to solve the corresponding non-routine problems on the first test. This allowed some distinction to be made between the lack of routine skills and the inability to access such skills in order to develop a solution method for the associated non-routine problem.

## 2. The Course, the Students, and the Tests

### 2.1 The Calculus / Differential Equations Sequence

The setting is a southeastern comprehensive state university having an engineering emphasis and enrolling about 7500 students -- the same university of the earlier studies of C and A/B first-term calculus students [12, 13]. The annual average ACT composite score of entering freshman is slightly above the national average for high school graduates, e. g., in the year the data was collected the university average was 21.1, compared to the national average of 20.6.

A large majority of students who take the calculus/differential equations sequence at this university are engineering majors. The rest are usually science or mathematics majors. A separate, less rigorous, three-semester-hour calculus course is offered for students majoring in other disciplines.

two tests and told he or she need not, in fact should not, study for them. The students were told that three groups of ten students would be randomly selected according to their first calculus grades (A, B, C), and in each group there would be four prizes of $20, $15, $10, and $5. The latter was an incentive to ensure that all students would be motivated to do their best. Altogether 11 A, 14 B, and 12 C students volunteered and ten were randomly selected from each group. Of those, 28 students (10 A, 9 B, and 9 C) actually took the tests: three mathematics majors and twenty-five engineering majors (nine mechanical, five chemical, four civil, four electrical, two industrial, and one undeclared engineering concentration). These majors reflected the usual clientele for the calculus/differential equations sequence.

At the time of the study, all but one of the 28 students tested had taken the third semester of calculus at this university; the one exception was enrolled in Calculus III and Differential Equations simultaneously. Their grades in Calculus III were 5 A, 8 B, 8 C, 4 D, and 2 F. Of these, one D student and one F student were repeating Calculus III while taking

In addition, five students had taken complex variables, and another was enrolled in that course at the time of the study. One of these five had also taken an introductory matrix algebra course, as had two other students. Except for one C, all grades for these students in these additional mathematics courses were at least B. In our analysis of the results we will compare the students who had studied mathematics beyond the calculus/differential equations sequence with those who had not.

Of the 28 students in this study, 22 (79%) graduated within six years of their admission to the university as freshmen. In comparison, for the university as a whole over the same time period, the average graduation rate within six years of admission was 41%.

As of May 1999, it is known that all but two of the 28 students tested had earned bachelor's degrees at this university, three in math and the others in engineering. In addition, it is known that five students had earned master's degrees in engineering and one had earned an MBA, all at this university. One student had earned a master's degree in mathematics at this university and a Ph.D. in mathematics at another university. There may be additional accomplishments of these kinds among the 28 students, but they could not all be traced.[2]

The students in this study represented 33% of the A's, 26% of the B's, and 6 % of the C's awarded in Differential Equations that semester, and none of the 30 D's, F's and W's. In addition, after a minimum of three semesters at this university, these 28 students had a mean GPA of 3.145 for all courses taken. Their graduation rate was almost double that of the university as a whole, and at least 25% of them went on to complete a graduate degree. By all these indicators, at the time of the study and subsequently, these students were among the most successful at the university.

### 2.3 The Tests

Students were allowed one hour to take the five-problem non-routine test, followed by half an hour for the ten-part routine test comprised of associated algebra and calculus exercises. Prior to the non-routine test, the students were told they might find some of the problems a bit unusual. No calculators were allowed. They were asked to write down as many of their ideas as possible because this would be helpful to us and to their advantage. They were told A students (in first calculus) would only be competing against other A students for prizes, and similarly, for B and C students. They were assured all prizes would be awarded and partial credit would be given.

Each non-routine problem was printed on its own page, on which student work was to be done. All students appeared to be working diligently for the entire hour. As in the

### 2.3.1 The Non-routine Test

1. Find values of $a$ and $b$ so that the line $2x+3y=a$ is tangent to the graph of $f(x)=bx^2$ at the point where $x=3$.
2. Does $x^{21} + x^{19} - x^{-1} + 2 = 0$ have any roots between –1 and 0? Why or why not?
3. Let $f(x)=\begin{cases} ax, & x \le 1 \\ bx^2+x+1, & x>1 \end{cases}$. Find $a$ and $b$ so that $f$ is differentiable at 1.
4. Find at least one solution to the equation $4x^3 - x^4 = 30$ or explain why no such solution exists.
5. Is there an $a$ such that $\lim_{x\to 3} \dfrac{2x^2-3ax+x-a-1}{x^2-2x-3}$

As soon as the non-routine tests were collected, the students were given the two-page

Thus, the differential equations students did perform somewhat better than the first-year calculus students.

### 3.2 Comparison of Non-Routine and Routine Test Results: Did Students Have Adequate Resources and Use Them?

The routine test was designed to determine whether the students' inability to do the non-routine problems was related to an inadequate knowledge base of calculus and algebra skills (i.e., "resources" [10, 11]). Did the students lack the necessary factual knowledge or did they have it without being able to access it effectively? Scores on the corresponding routine questions (Table 3) were taken as indicating the extent of a student's factual knowledge regarding a particular nonroutine problem. As in [13], a student was considered to have *substantial factual knowledge* for solving a nonroutine problem if that student scored at least 66% on the corresponding routine questions. A student was considered to have *full factual knowledge* for solving a nonroutine problem if that student's answers to the corresponding routine questions were correct, except possibly for notation, for example, answering (1, -1) instead of -1 to Question 6. All others were considered as having less than substantial factual knowledge.

In Table 4 we give the number of students who exhibited substantial or full factual knowledge on the corresponding routine questions with the number of students whose solution to a particular non-routine problem was completely correct or substantially correct. For example, on Problem 1, 15 (of 28) students had full factual knowledge; of these 5 gave completely correct and 2 gave substantially correct solutions. An additional ten (of 28) students had substantial factual knowledge for Problem 1, but none of them gave completely or substantially correct solutions. That is, the performance of these ten students on the routine questions seemed to indicate they had sufficient factual knowledge to solve, or at least make substantial progress on, Problem 1; yet they either did not access it or were unable to use their knowledge effectively to make progress. The remaining three students demonstrated less than substantial factual knowledge, making a total of 7 students giving completely or substantially correct solutions on Problem 1.

Taking another perspective, in the 59 routine question solution attempts in which students demonstrated full factual knowledge, they were able to solve the corresponding non-routine problem 14 times (24%) or make substantial progress towards its solution 6 times (10%). Thus, on slightly more than a third of their attempts (34%), these students accessed their knowledge effectively. Students with substantial, but not full factual knowledge, did so on less than a quarter of their attempts. These results are summarized in Table 5. Furthermore, six students showed *T*

| | Non-Routine Problem 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Full Factual Knowledge** | 15 | 3 | 5 | 14 | 22 |
| Non-Routine Problem | | | | | |
| Completely Correct | 5 | 1 | 1 | 0 | 7 |
| Substantially Correct | 2 | 0 | 1 | 0 | 3 |
| **Substantial Factual Knowledge** | 10 | 19 | 12 | 1 | 2 |
| Non-Routine Problem | | | | | |
| Completely Correct | 0 | 2 | 4 | 0 | 0 |
| Substantially Correct | 0 | 3 | 1 | 0 | 0 |
| **Less than Substantial Factual Knowledge** | 3 | 6 | 11 | 13 | 4 |
| Non-Routine Problem | | | | | |
| Completely Correct | 0 | 0 | 0 | 0 | 0 |
| Substantially Correct | 0 | 0 | 1 | 0 | 1 |

| | Full Factual Knowledge (59) | Substantial, but not full Factual Knowledge (44) |
|---|---|---|
| Completely correct non-routine solution | 24% (14/59) | 14% (6/44) |
| Substantially correct non-routine solution | 10% (6/59) | 9% (4/44) |

TABLE 5. Overview, giving the percentage of correct solutions by those having the requisite factual knowledge

In order to compare overall student performances on the routine and non-routine tests, we introduce the notion of a *score pair*, denoted $\{a, b\}$, where $a$ is the student's score on the routine test and $b$ is the student's score on the non-routine test. In every case, $a > b$. Figure 1 shows students' routine test scores in descending order (from left to right); superimposed below each student's routine test score is her/his non-routine test score.

The three students with the highest non-routine scores had score pairs of {90, 69}, {85, 59}, and {83, 59}; the first of these subsequently obtained a B.S. in civil engineering, summa cum laude with a cumulative grade point average of 3.948. The three mathematics majors in the study, all of whom had completed at least one additional mathematics course at the time of the study, had score pairs {95, 34}, {89, 18} and {87, 21}. That is, they scored in the top quarter on the routine test, but the latter two scored slightly below than the mean non-routine score of 21.3. The last of these three subsequently obtained a Ph.D. in mathematics from a major state university.

Student performance on the routine and non-routine tests was not improved by having studied additional mathematics. The respective mean scores for the eleven students who had done so were 73.1 (vs. 75.3 for all of the students) and 15.3 (vs. 21.3 for all of the students).

Figure 1 shows a generally positive relationship between factual knowledge (resources) and the ability to solve novel problems. Nonetheless, having the resources for a particular problem is not enough to assure that one will be able to solve it. Two students had score pairs of {86, 4} and {80, 3}suggesting that having a reasonably good knowledge base of calculus and algebra skills (resources) is not sufficient to make substantial progress on novel calculus problems. One student, score pair {83, 59}, lacked substantial factual knowledge on only those routine questions associated with Problems 2 and 4 (on which he scored zero)

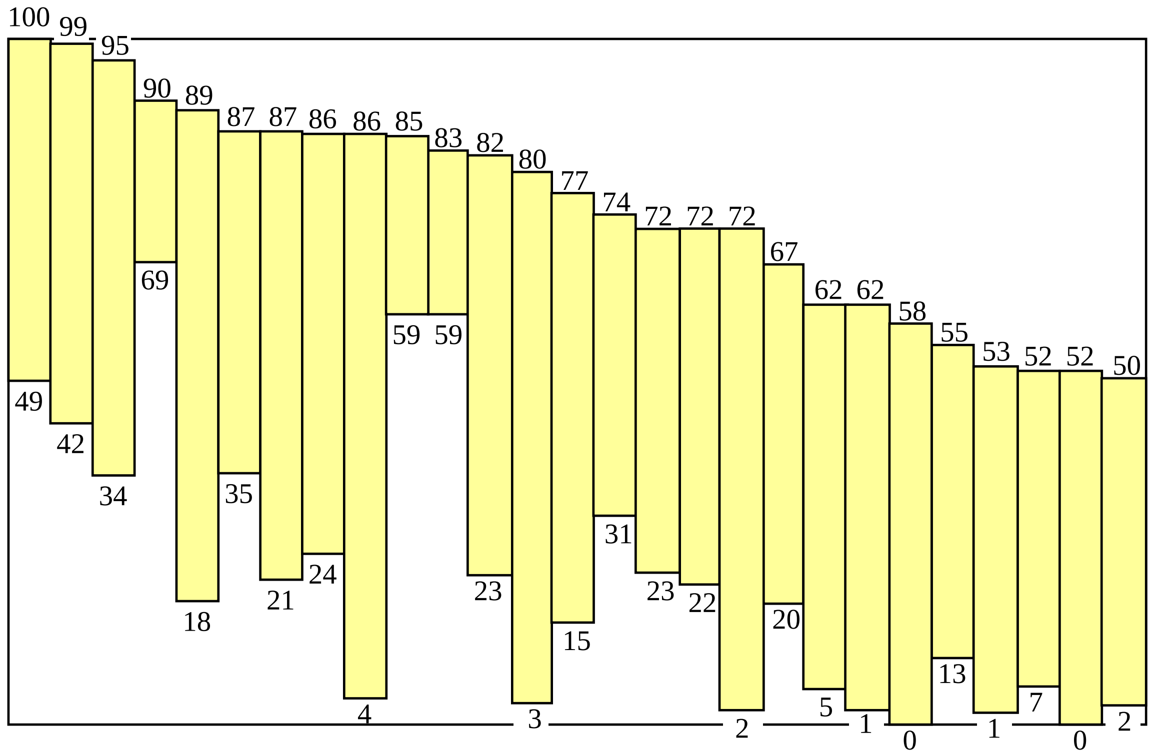


FIGURE 1. Upper score is for routine / factual knowledge test.
Lower score is for non-routine test.

## 4. Favored Solution Methods

### 4.1 Non-routine Problem 1.

*Find values of $a$ and $b$ so that the line $2x+3y=a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x=3$.*

Fifteen students set the equation of the line equal to that of the parabola and solved for either $a$ or $b$. Eight of these ignored the tangency of the line to the curve. The remaining seven also set the derivative of $f$ equal to the slope of the line to obtain a second equation so $a$ and

students, this approach was also used by 20% of the students. Only two students used the method favored by C calculus students in the earliest study [12]: factor $4x^3 - x^4$ and set each factor equal to 30. Four students took the first derivative, set it equal to zero and stopped. Several students used synthetic division to check whether particular values were roots.

### 4.5 Non-routine Problem 5.

*Is there an a such that* $\lim_{x \to 3} \frac{2x^2 - 3ax + x - a - 1}{x^2 - 2x - 3}$ *exists? Explain your answer.*

On Problem 5, there were 39% (11 of 28) substantially or completely correct solutions, whereas for the A/B calculus students, 37% (7 of 19) made substantial progress towards a solution. Of the 28 attempts, 15 involved the use of L'Hôpital's Rule. Nine of these attempts were at least partially successful. In the other six instances students failed to note that the numerator as well as the denominator must have limit zero before applying L'Hôpital's Rule. Five students substituted 3 for $x$, found the denominator of the expression to be zero and asserted that no limit could exist since the denominator was zero at the limiting value of the variable (two of these were math majors). This was the favored method of the C calculus students (47% of them used it) and was used by 26% of the A/B calculus students [12, 13]. Here it was found in just 18% (5 of 28) of the solution attempts. Six students struggled, algebraically, with finding a way to factor the numerator so that the ($x$-3) in the factored denominator could be cancelled and the limit taken; two of these resulted in completely correct solutions.

### 4.6 Summary

#### 4.6.1 Use of calculus

Where students who successfully solved Problem 1 in the present study were willing to draw and reject graphs, those in the A/B study who chose to use graphs generally had only one -- either it was the right one and the solution was substantially or completely correct or it was an incorrect graph and the solution was incorrect. In the A/B study, 16 graphs were produced by 12 students on Problem 1 whereas in the present study, 37 graphs were drawn by 22 students. Three of the 12 A/B students who used graphs (25%) rejected (crossed out) one of their graphs (including one who rejected a graph which was correct) while six of the 22 (27%) who used graphing on Problem 1 in this study rejected a graph. It would appear, then, that though the more experienced students were more willing to posit graphical ideas (81% versus 63%) than the less experienced students they were about equally likely to reject the graphs they produced.

DeFranco's paper on expert problem solvers with Ph.D.s in mathematics suggests that the skills possessed by the experts which are often lacking in the non-experts might include a willingness to create *and abandon* (reject) *and revisit* many ideas in the solution process [5]. Thus it might be useful to know how students develop the willingness to risk commiting graphical and other ideas to paper and to reject such ideas once they have been given life on paper.

## 5. Analysis

Reflecting on the three studies, one wonders when, if ever, students learn to use calculus flexibly enough to solve more than a few nonroutine problems. Furthermore, how could it happen that students, who were more successful than average by a variety of traditional measures and who demonstrated full factual knowledge for a non-routine problem, failed to access and use their knowledge successfully on 76% of their attempts (Table 5)? Many of these students (the engineering majors) had almost completed their formal mathematical educations, except possibly for one or two upper-division mathematics courses, leaving them limited opportunity in future mathematics courses to improve their non-routine problem-solving abilities. Finally, does it matter whether students can solve such non-routine problems?

### 5.1. When do students finally learn to apply calculus flexibly?

It would appear from this sequence of three studies that, at least for traditionally-taught calculus students in classes of 35-40 students, the ability to solve non-routine calculus problems develops only slowly. Performance for the best students went from one third who could solve at least one non-routine beginning calculus problem toward the end of their first year of college calculus to slightly less than half who could do so toward the end of the two-year calculus/differential equations sequence. In addition, the percentage of correct solutions increased modestly over the three studies (Table 6).

On the other hand, by the time these students were coming to the end of their calculus/differential equations sequence their algebra skills seem to be relatively sophisticated and readily accessible, albeit somewhat flawed. Such slow, incremental

| Study | Completely Correct Solutions | Substantially Correct Solutions |
|---|---|---|
| DE | 14% (20/140) | 9% (12/140) |
| (A/B) Calculus | 9% ( 9/95) | 9% ( 9/95) |
| (C) Calculus | 0% ( 0/85) | 7%( 6/85) |

TABLE 6. Percent of correct solution attempts in all three studies

### 5.2 How does it happen that students can have the knowledge, but not be able to effectively use it to solve non-routine problems?

Part of the rationale for having the students take the routine test after the non-routine test was to determine whether they had the requisite algebra and calculus skills to solve the non-routine problems, but were unable to think of or use them; this was a concern raised by process-oriented research. A person who has reflected on a number of problems is likely to have seen (perhaps tacitly) similarities between some of them. He or she might be regarded as recognizing (not necessarily explicitly or consciously) several overlapping problem situations, each arising from problems with similar features. For example, after much exposure many students will recognize a problem as one involving factoring, several linear equations, or integration by parts, etc.[4] Such problem situations act much like concepts. While these situations may lack names, for a given individual they are likely to be associated with images, i.e., strategies, examples, non-examples, theorems, judgments of difficulty, and the like. Following Tall and Vinner's idea of concept image [16], we will call this kind of mental structure a *problem situation image* and suggest that some such images may, and others may not, contain what we will call *tentative solution starts*: tentative general ideas for beginning the process of finding a solution. The linking of problem situations with one or more tentative solution starts is a kind of (perhaps tacit) knowledge. For instance, the image of a problem situation asking for the solution to an equation might include "try getting a zero on one side and then factoring the other." It might also include "try writing the equation as $f(x) = 0$ and looking for where the graph of $f(x)$ crosses the $x$-axis," or even "perhaps the max of $f$ is negative so $f(x) = 0$ has no solution." We suggest that the problem-solving process is likely to include the recognition of a problem as belonging to one or more problem situations, and hence, to bring to mind a tentative solution start contained in one's image of one of those problem situations. This may, in turn, mentally prime the recall of resources from one's knowledge base. Thus a tentative solution start may link recognition of a problem situation with recall of appropriate resources, i.e., what we have called full factual knowledge. For example, in this study a number of students tried factoring on Problem 4. When this approach did not work, had those students' images of such problems included "try looking at whether the graph crosses the x-axis," they might have recalled their knowledge of graphs and calculus to discover that the maximum was too small for the equation to be solvable. In viewing our data from this perspective, we are suggesting that problem situations, their images, and the associated tentative solution starts all vary from student to student and that the process of mentally linking recognition (of a problem situation) to recall (of requisite resources) through a problem situation image might occur several times in solving a single problem. We are not suggesting this is the only way access might occur, just that it could play a role in solving the kind of moderately non-routine problems discussed in this paper.

Except while taking tests the students in this study would have had worked examples (from textbooks and lectures notes) available to them during most of their previous problem-solving attempts. Those who habitually consulted such worked examples before attempting their own solutions would have had little occasion to reflect on multiple tentative solution starts. Such students might well have problem situation images with few tentative solution starts, thereby reducing the usefulness of their situation images in priming the recall of their factual knowledge; this could happen even if they realized a new idea is needed, that is, even if they did not lack (metacognitive) control [10, 11]. In summary, we are suggesting that some of the students who were unable to solve non-routine problems while having full factual knowledge may have lacked a particular kind of knowledge, namely tentative solution

[4] Although the kinds of features noticed by students in mathematical problem situations do not seem to have been well studied, the features focused on in physics problem situations have been observed to correspond to degree of expertise. Novices tend to favor surface characteristics (e.g., pulleys), whereas experts tended to focus on underlying principles of physics (e.g., conservation of energy). [4]

starts, and this might be due to a combination of the way homework exercises had been presented and the students' study habits.

Much of our data concerning those students who had adequate factual knowledge, but did not access it to solve non-routine problems, is consistent with the above analysis. However, our data do not adequately encode the problem-solving process for this proposed explanation to be more than a conjecture which might be examined in a future study.

### 5.3 Does it matter whether students are able to solve non-routine problems?

Perhaps surprisingly, the answer seems to be both yes and no. No, because the students in this study were among the most successful at the university by a variety of traditional indicators, both at the time of the study and subsequently, yet half of them could not solve a single non-routine problem. They had overall GPAs of just above 3.0 at the time of the study and almost double the graduation rate of the university as a whole. At least seven of them subsequently earned a master's degree and one a Ph.D. in mathematics. Furthermore, the idea that traditional success may not require very much non-routine problem-solving ability, including metacognitive control, is supported by De Franco's problem-solving study comparing mathematicians of exceptional creativity (e.g., Fields medallists) with very successful published Ph.D. mathematicians. He found that while both were content experts, only the former were problem-solving experts [4]. Thus, it seems possible to be academically successful in mathematics and related subjects without being able to consistently solve non-routine problems, especially the more difficult ones in which Schoenfeld's problem-solving characteristics (heuristics, control, beliefs [10, 11]) play a large role.

not requiring, for example, the consideration of multiple sub-problems. The non-routine test in this study consisted of problems of this kind. Finally, at the opposite end of the continuum from routine problems, one might place *very non-routine* problems which, while dependent on knowledge of the course, may involve more insight, the consideration of several sub-problems or constructions, and use of Schoenfeld's behavioral problem-solving characteristics (heuristics, control, beliefs [10, 11]). For such problems a large supply of tentative solution starts, built up from experience, might not be adequate to bring to mind the knowledge needed for a solution, while for moderately novel problems it probably would. Often the Putnam Examinations [17] include such very non-routine problems.[5]

Most university mathematics teachers would probably like students who pass their courses to be able to work a wide selection of routine, or even moderately routine, problems. In addition, we believe that many such teachers would expect their better students to be able to work moderately non-routine problems, and think of this as functionally equivalent to having a good conceptual grasp and understanding of the course. In other words, the ability to dents7phct,entsequivalent ch to7a f9 3c"g of ththe

## REFERENCES


1. Amit, M. and Vinner, S., *Some misconceptions in calculus: Anecdotes or the tip of an iceberg?* In G. Booker, P. Cobb, and T. N. de Mendicuti (Eds.), Proceedings of the 14th International Conference of the International Group for the Psychology of Mathematics Education, 1 (1990), CINVESTAV, Mexico, pp. 3-10.
2. Berkey, D. P., *Calculus*, 2nd ed., Saunders College Publishing, New York, 1988.
3. Carlson, M. P., *A cross-sectional investigation of the development of the function concept*, CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education III, 7 (1998), 114-162.
4. Chi, M. T. H., Feltovich, P., and Glaser, R., *Categorization and representation of physics problems by experts and novices*, Cognitive Science 5 (1981), 121-152.
5. DeFranco, T. C., *A perspective on mathematical problem-solving expertise based on the performance of male Ph.D. mathematicians*, CBMS Issues in Mathematics Education: Research in Collegiate Mathematics Education II, 6 (1996), 195-213.
6. o p t d 0 . S